Does the gluon spin contribute in a gauge-invariant way to nucleon spin?

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Although the matrix element of the gluon spin operator in nucleon helicity states is known to be independent of some special choices of gauge, we show that it is not invariant under a general gauge transformation. We find that there exists a simple means of obtaining the matrix element in a different choice of gauge from a calculation made in one specific gauge. Similar conclusions hold for other manifestly gauge-dependent operators present in the QCD angular momentum operator. [S0556-2821(99)05123-1]

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In quantum chromodynamics (QCD), the gluon spin operator, $\vec{S}_g$, can be defined as

$$\vec{S}_g = \int d^3x \vec{E} \times \vec{A}, \tag{1}$$

where we have suppressed the color indices, $\vec{E}$ is the color electric field and $\vec{A}$ is the gauge potential. The above operator is manifestly gauge dependent, and therefore one expects that its matrix element in a physical state depends on the gauge choice. Indeed, in Ref. [1] it was shown that the matrix element of $\vec{S}_g$ in a quark helicity state is different in the axial ($A^+=0$) and covariant gauges. On the other hand, it was claimed in Ref. [2] that this matrix element in a helicity state is invariant under a gauge transformation. The goal of this paper is to clarify the relationship of the two results. In particular, we point out that the proof in Ref. [2] covers only a special class of gauge transformations and the matrix element of $\vec{S}_g$ *is* in fact gauge dependent. We then make a number of pertinent observations about comparing calculations in different gauges. In particular, we demonstrate that there exists a simple means of getting the answer in a different choice of gauge from calculations made in one specific gauge.

We start with general comments about calculations in quantum gauge theories. Because of the gauge symmetry and the infinite number of degrees of freedom in fields, it is always necessary to choose a gauge for calculating Green's functions and physical matrix elements, perhaps with the exception of lattice gauge theory in which fields are assigned at a finite number of spacetime points. Indeed, in a canonical quantization of gauge theories, one has to choose a gauge at the very outset. In the path-integral formulation, a gauge choice is conveniently made by selecting a set of gauge conditions,

$$F^a(A) = \sigma^a(x), \tag{2}$$

where the index a runs over the number of generators of the gauge group (and will be omitted in the remaining equations). In addition, one can choose an arbitrary weighting functional $G(\sigma)$ to integrate over the auxiliary field σ. Without loss of generality, one can assume $\int [D\sigma] G(\sigma) = 1$, where D denotes the functional integration measure. Hence, the generating functional for Green's functions in QCD reads [3],

$$Z(J) = \frac{1}{\mathcal{N}} \int [D\phi][D\sigma] \Delta_F(A) \delta(F(A) - \sigma) G(\sigma) \times \exp\left(iS + i \int d^4x [J \cdot A + \bar{\eta}\psi + \bar{\psi}\eta] \right), \tag{3}$$

where ϕ denotes a collection of fields A, ψ, and $\bar{\psi}$, and S is the canonical QCD action,

$$S = \int d^4x \left[\bar{\psi}(i\not{D} - m_q)\psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right], \tag{4}$$

and the normalization constant $\mathcal{N}$ is

$$\mathcal{N} = \int [D\phi][D\sigma] \Delta_F(A) \delta(F(A) - \sigma) G(\sigma) e^{iS}, \tag{5}$$

so that $Z[0]=1$. The Faddeev-Popov determinant $\Delta_F(A)$ is defined such that

$$\Delta_F(A) \int [D\omega] \delta(F({}^{\omega}A) - \sigma) = 1, \tag{6}$$

where ${}^{\omega}A$ is a gauge transformation of A and $[D\omega]$ integrates over the gauge group at every point of spacetime.

To calculate the physical matrix element $\langle f | \hat{O}(\phi) | i \rangle$ of an operator $\hat{O}(\phi)$, one can start with the following Green's function:

$$\langle 0 | T[J_f(x) O(y) J_i^\dagger(0)] | 0 \rangle |_{F,G} = \frac{1}{\mathcal{N}} \int [D\phi][D\sigma] \Delta_F(A) \delta(F(A) - \sigma) G(\sigma) J_f(\phi) O(\phi) J_i^\dagger(\phi) e^{iS}, \tag{7}$$

where $J_{i,f}$ are the interpolation fields or currents for the initial and final physical states. According to the Lehmann-

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Symanzik-Zimmermann reduction formula [3], $\langle f|\hat{O}(\phi)|i\rangle$ is just the residue of the Green's function at the poles corresponding to external physical states, modulo the coupling constants $Z_{i,f}$ defined as $\langle 0|J_{i,f}(0)|i,f\rangle=\sqrt{Z_{i,f}}$ (where a possible Lorentz structure is suppressed). Gauge invariance of the physical matrix element means that the residue depends on a choice of $F(A)$ and $G(\sigma)$ only through the couplings $Z_{i,f}$ when the interpolating currents are gauge dependent.

Using methods available in textbooks [3], one can show that the Green's function $\langle 0|T[J_f(x)O(y)J_i^\dagger(0)]|0\rangle|_{F,G}$ is independent of the choice of $F(A)$ and $G(\phi)$ provided that $J_i(\phi)$, $J_f(\phi)$, and $O(\phi)$ are gauge invariant. Indeed, consider a different choice of $\tilde{F}$ and $\tilde{G}$ such that

$$\Delta_{\tilde{F}}(A)\int[D\omega]\delta(\tilde{F}({}^{\omega}A)-\tilde{\sigma})=1;\qquad \int[D\tilde{\sigma}]\tilde{G}(\tilde{\sigma})=1. \tag{8}$$

Multiplying both unities to the right-hand side of Eq. (7), we get

$$\begin{aligned}
&\langle 0|T[J_f(x)O(y)J_i^\dagger(0)]|0\rangle|_{F,G}\\
&=\frac{1}{\mathcal{N}}\int[D\phi][D\sigma][D\tilde{\sigma}][D\omega]\Delta_F(A)\delta(F(A)-\sigma)G(\sigma)J_f(\phi)O(\phi)J_i^\dagger(\phi)\times\Delta_{\tilde{F}}(A)\tilde{G}(\tilde{\sigma})\delta(\tilde{F}({}^{\omega}A)-\tilde{\sigma})e^{iS}\\
&=\frac{1}{\mathcal{N}}\int[D\phi][D\sigma][D\tilde{\sigma}][D\omega]\Delta_F(A)\delta(F({}^{\omega^{-1}}A)-\sigma)G(\sigma)J_f({}^{\omega^{-1}}\phi)O({}^{\omega^{-1}}\phi)\times J_i^\dagger({}^{\omega^{-1}}\phi)\Delta_{\tilde{F}}(A)\tilde{G}(\tilde{\sigma})\delta(\tilde{F}(A)-\tilde{\sigma})e^{iS}\\
&=\frac{1}{\mathcal{N}}\int[D\phi][D\sigma][D\tilde{\sigma}]G(w)J_f({}^{\omega_0}\phi)O({}^{\omega_0}\phi)J_i^\dagger({}^{\omega_0}\phi)\Delta_{\tilde{F}}(A)\tilde{G}(\tilde{\sigma})\delta(\tilde{F}(A)-\tilde{\sigma})e^{iS}\\
&=\frac{1}{\mathcal{N}}\int[D\phi][D\tilde{\sigma}]J_f(\phi)O(\phi)J_i^\dagger(\phi)\Delta_{\tilde{F}}(A)\tilde{G}(\tilde{\sigma})\delta(\tilde{F}(A)-\tilde{\sigma})e^{iS},
\end{aligned} \tag{9}$$

where after the second equal sign, we have made the gauge transformation $\phi\to{}^{\omega^{-1}}\phi$ and used the fact that the measure $[D\phi]$ and determinants $\Delta_F(A)$ and $\Delta_{\tilde{F}}(A)$ are invariant under the transformation. After the third equal sign, we have integrated out the $[D\omega]$ by using the δ-function constraint $F({}^{\omega^{-1}}A)=\sigma$. For a fixed A and σ, there is a special ω_0 that fulfills the constraint, and hence ω_0 is a specific function of them: $\omega_0=\omega^{-1}(A,\sigma)$. After the fourth equal sign, we have used the assumption that $J_{i,f}(\phi)$ and $O(\phi)$ are gauge invariant. This makes the functional integration over σ trivial. The final line establishes the gauge independence of the Green's function: $\langle 0|T[J_f(x)O(y)J_i^\dagger(0)]|0\rangle|_{F,G}=\langle 0|T[J_f(x)O(y)J_i^\dagger(0)]|0\rangle|_{\tilde{F},\tilde{G}}$.

From the above discussion, we see that gauge invariance of a matrix element is guaranteed when the observable under consideration, defined as a functional of physical fields in the path integral formalism, is invariant under *any* gauge transformation ω, including arbitrary dependence on the gauge potential A itself. The notion of A-dependent gauge transformation can be found in textbooks, but it is usually discussed in the context of classical gauge theory. In a canonically quantized theory, an A-dependent ω can no longer be considered as a transformation ''parameter,'' but rather a quantum operator in Hilbert space. Because the commutation relations between ω and the fundamental fields depend on the choice of transformation itself, it is difficult to consider the most general gauge transformations in canonically quantized theories. In fact, as we have alluded to before, canonical quantization of a gauge theory can only be carried out after a choice of gauge, and different gauge choices often lead to different quantum Hilbert spaces. When comparing calculations in different gauges, one usually compares the final results only—not the actual physical states and operators in those gauges. In particular, we know of no discussion in the literature about how to transform a quantum operator from one gauge to another.

Fortunately, in the path-integral formulation, all field variables are treated as classical, and a transformation from one gauge to another, performed with an A-dependent ω, can be easily handled by a change of integration measure and the associated Jacobian (the Faddeev-Popov determinant). A decisive test of the gauge invariance of an observable can be done with a general gauge transformation: an $\omega(A)$ solving $F({}^{\omega}A)=\sigma$ for any A satisfying $\tilde{F}(A)=\tilde{\sigma}$. It can happen that certain matrix elements are invariant under a special class of gauge transformations; in particular, those that keep the Faddeev-Popov determinant invariant. However, if they cannot pass the general test above, one cannot claim their total gauge independence.

In Ref. [2], Chen and Wang claim to have shown that the matrix element of $\vec{S}_g$ in a helicity eigenstate is gauge invariant. What they have actually shown is that the matrix element is the same in gauges $F(A)=\sigma$ and $F({}^{\omega}A)=\sigma$, where ω is an A-independent gauge transformation parameter. An outline of the proof goes as follows (using gauge-invariant interpolating operators for external states):

$$\begin{aligned}\langle 0|T[J_f(x)S_g(y)J_i^\dagger(0)]|0\rangle|_{F({}^\omega A),G} &= \frac{1}{\mathcal{N}}\int [D\phi][D\sigma]\Delta_F(A)\delta[F({}^\omega A)-\sigma] \\ &\quad\times G(\sigma)J_f(\phi)S_g(\phi)J_i^\dagger(\phi)e^{iS} \\ &= \frac{1}{\mathcal{N}}\int [D\phi][D\sigma]\Delta_F(A)\delta[F(A)-\sigma] \\ &\quad\times G(\sigma)J_f(\phi)S_g({}^{\omega^{-1}}\phi)J_i^\dagger(\phi)e^{iS},\end{aligned} \tag{10}$$

where after the second equality one has made a gauge transformation ω^{-1}, and all the ω dependence now appears in S_g. We write

$$S_g({}^{\omega^{-1}}\phi)\equiv S_g(\phi)+\delta S_g(\phi,\omega). \tag{11}$$

Chen and Wang showed that $\delta S_g(\phi,\omega)$ has a zero matrix element in a helicity state and so

$$\begin{aligned}\langle 0|T[J_i(x)S_g(y)J_f^\dagger(0)]|0\rangle|_{F({}^\omega A),G} &= \langle 0|T[J_i(x)S_g(y)J_f^\dagger(0)]|0\rangle|_{F(A),G}.\end{aligned} \tag{12}$$

As we have demonstrated above, Eq. (12) is not sufficient to guarantee that that matrix element remains invariant under an arbitrary new gauge condition $\tilde{G}(A)=\tilde{\sigma}$. Indeed, under A-dependent gauge transformations, the proof in Ref. [2] no longer applies.

As an example of A-dependent gauge transformation, we compare calculations in the covariant gauge and light-cone gauge $(A^+=0)$. The vacuum matrix element of a general operator $\hat{O}$ (which can be a time-ordered product of operators at several different spacetime points) in the covariant gauge can be written as

$$\begin{aligned}\langle 0|\hat{O}|0\rangle|_L &= \frac{1}{\mathcal{N}}\int [D\phi][D\sigma]\Delta_F(A)\delta(\partial\cdot A-\sigma)O(\phi) \\ &\quad\times\exp\left(-i\int d^4x\sigma(x)^2/2\lambda\right)\exp(iS),\end{aligned} \tag{13}$$

where we have included the standard Gaussian weighting function. The same matrix element in light-cone gauge $A^+=0$ can be expressed as

$$\langle 0|\hat{O}|0\rangle|_A=\frac{1}{\mathcal{N}}\int [D\phi]\Delta_F(A)\delta(A^+)O(\phi)\exp(iS). \tag{14}$$

Using the same method used in deriving Eq. (9), we can show that the matrix element in two different gauges is related by the following equation:

$$\langle 0|\hat{O}(\phi)|0\rangle|_A=\langle 0|\hat{O}({}^\omega\phi)|0\rangle|_L, \tag{15}$$

where ω is a gauge transformation which brings a gauge configuration to the $A^+=0$ gauge. In perturbation theory where the coupling g is small

$$\omega^a=-\frac{1}{\partial^+}A^{a+}+\mathcal{O}(g). \tag{16}$$

Equation (15) is both interesting and important—it relates any matrix element (or Green's function) in the axial gauge to that of a gauge-transformed operator in the covariant gauge. For instance, using the relation one can easily recover the gluon or quark field propagators in the light-cone gauge from those in the covariant gauge. We emphasize here that similar relations between any two independent gauges can be derived and they are entirely *nonperturbative*. The relationship between the covariant and axial gauge calculations has also been studied recently by Joglekar and Misra [4]. In those studies, generalized BRST transformations are used to connect different gauge choices. Some other references for tranformation between different gauges can be found in [5].

As an application of Eq. (15), we consider the difference of the $\vec{S}_g$ matrix elements in the covariant and light-cone gauges,

$$\langle f|S_g|i\rangle|_A-\langle f|S_g|i\rangle|_L=\langle f|S_g({}^\omega A)-S_g(A)|i\rangle|_L, \tag{17}$$

which can be calculated completely in the covariant gauge. Given ω in Eq. (16), one cannot show, perturbatively or nonperturbatively, that the right-hand side vanishes when the external states have definite helicity. In fact the above equation can be used to directly check the one-loop calculation presented in Ref. [1]. Consider an ''on-shell'' quark in the state of momentum p^μ and helicity 1/2. We notice that the quark fields are not gauge invariant. However, the difference between the original and the gauge-transformed quark fields has no perturbative quark pole; therefore, Eq. (17) still applies. At one-loop order we find

$$\left\langle p+\frac{1}{2}\middle|\delta S_g(A)\middle|p+\frac{1}{2}\right\rangle=C_F\frac{\alpha_s}{2\pi}\ln\left(\frac{Q^2}{\mu^2}\right), \tag{18}$$

where $\delta S_g=S_g({}^\omega A)-S_g(A)$, $C_F=(N_c^2-1)/(2N_c)$ with N_c being the number of colors, and Q^2 and μ^2 are the ultraviolet and infrared cutoffs, respectively. The above result is exactly what we found in Ref. [1].

We have also found that in the one-loop calculation the $\vec{S}_g$ matrix element in the covariant gauge is independent of the Feynman parameter λ. By studying the relations between matrix elements in $\partial^\mu A_\mu=\sigma$ and $\partial^\mu A_\mu=\sigma/\sqrt{\lambda}$ gauges, we can show the λ independence directly. Notice that the different choice in λ, like the gauge transformations considered in Ref. [2], leaves the Faddeev-Popov determinant invariant.

To summarize, we demonstrated that the general gauge independence of a physical matrix element must be checked in the path-integral formalism through gauge-field-dependent gauge transformations. Using this, we showed that the gluon spin contribution to the nucleon spin is indeed gauge dependent. The conclusion also applies readily to other gauge-dependent operators considered in Ref. [2]. We derived a general relation between matrix elements in the covariant

and axial gauges. Using the relation, we calculated the one-loop difference of the gluon spin contribution to the quark spin in the two gauges. The result confirms the explicit calculation presented in Ref. [1].

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