Novel approach to decays, gluon distributions, and fragmentation functions of heavy quarkonia

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An effective, low-energy, field theory of s-wave quarkonia, constituent heavy quarks, and gluons is constructed which is manifestly gauge invariant. The interaction Lagrangian has the form of a twist expansion, as typically encountered in hard processes, and involves derivatives of arbitrary order. The parameters in the interaction are related with the nonrelativistic wave function, and standard results for $Q\bar{Q}$ inclusive decays and radiative transitions are shown to be easily recovered. The light-cone gluon momentum distribution at very small x is calculated and shown to be uniquely determined by the nonrelativistic wave function. The distribution has a part which goes as $x^{-1}\ln x$, i.e., is more singular than the usually assumed $1/x$ behavior. The fragmentation function for a virtual gluon to inclusively decay into an η_c or η_b is also calculated. We find that the emission of low momentum gluons makes this process quite sensitive to assumptions about the binding energy of heavy quarks in quarkonia.

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I. INTRODUCTION

Heavy quarkonium is traditionally modeled as a nonrelativistic, color singlet bound state of a $Q\bar{Q}$ pair with a static Coulomb potential at short relative distances, and some confining type of potential at long distances. The binding energy, which is small relative to the heavy quark masses, is taken as justification for low quark relative velocity as well as the neglect of explicit gluon degrees of freedom in the hadron's wave function. In hard processes, the entire nonperturbative QCD physics is buried into a single parameter: the wave function at the origin for the case of S-wave quarkonia, or its derivative for the case of P waves. This model has been widely used in calculating the decay rates of quarkonium states, as well as their production in e^+e^- collisions, deep inelastic processes, Z^0 decays, etc. [1,2].

The problem with this traditional approach to quarkonium modeling is that gauge invariance, which is obviously fundamental to QCD because it is its legitimizing principle, is not respected as an exact symmetry. Under a local gauge transformation $q(\mathbf{x},t) \to U(\mathbf{x},t)q(\mathbf{x},t)$, the state normally used to describe quarkonia,

$$\int d^3x_1 d^3x_2 f(\mathbf{x}_1 - \mathbf{x}_2)\bar{q}(\mathbf{x}_1,t)\Gamma q(\mathbf{x}_2,t)|0\rangle \tag{1}$$

does not remain invariant. In the above equation Γ is a space-time-independent matrix in spin, color, and flavor indices and $f(\mathbf{x})$ is the relative wave function. However, one can construct gauge-invariant states in the following manner [3]: define

$$Q(\mathbf{x},t) = U^{-1}(\mathbf{x},t)q(\mathbf{x},t) \ , \tag{2}$$

$$U(\mathbf{x},t) = P\exp\left(ig\int_0^{\mathbf{x}} d\mathbf{y}\cdot\mathbf{A}(\mathbf{y},t)\right) \ . \tag{3}$$

Then, the state constructed from Q and $\bar{Q}$,

$$\int d^3x_1 d^3x_2 f(\mathbf{x}_1 - \mathbf{x}_2)\bar{Q}(\mathbf{x}_1,t)\Gamma Q(\mathbf{x}_2,t)|0\rangle \tag{4}$$

is indeed invariant for arbitrary $f(\mathbf{x})$. The gluon field A^μ transforms in the usual way:

$$A^\mu \to U^{-1}A^\mu U - \frac{i}{g}U^{-1}\partial^\mu U \ . \tag{5}$$

Unfortunately the operators $Q(\mathbf{x},t)$ are not pure (current) quark fields; they also involve arbitrary numbers of soft gluons which are responsible for transporting color between the quarks and for quarkonium binding. These constituent quarks are clearly extremely complicated objects. Therefore, to make a gauge-invariant model of quarkonium requires more than that suggested by Eq. (1). Fortunately, traditional quarkonium models are not widely off the mark because quarkonia are fairly small and the path-ordered integral in Eq. (4) is possibly negligible. For p waves one expects that the problem would be more acute than for s waves since a centrifugal barrier serves to keep the quarks apart, leading to a larger meson. However, to our knowledge, the validity of using a nongauge invariant state for either s or p systems has not been investigated.

II. THE MODEL

Heavy quarks can be considered as external sources placed in a gluonic vacuum which undergoes nonperturbative fluctuations, and results in a modification of the potential-type interaction between quarks in quarkonium [4]. For a large quark mass m, one can consider a $Q\bar{Q}$ pair localized at a relative distance R such that $m \gg 1/R$, and hence the relative momentum of quarks $p \ll m$. This together with the assumption that the quarkonium system is weakly bound, ensures that the nonrelativistic approximation is valid. At the same time, we would like perturbative methods to be applicable, i.e., $\alpha_s(p) \ll 1$, and hence that $R \ll 1/\Lambda_{\rm QCD}$. We shall assume that charmonium systems (bottomonium is obviously better) fulfill the requirement of being sufficiently heavy, yet also sufficiently small and weakly bound.

We would like to construct an effective theory of quarkonia, constituent quarks, and gluons which respects gauge invariance. Towards this end, consider an elementary pseudoscalar meson field $\phi(x)$, representing an η_c or η_b meson for example, which interacts with quark fields $Q(x)$ according to $\bar{Q}\gamma_5 Q\phi$. This is clearly wrong since quarkonia are extended, weakly bound, systems which can be formed only when the heavy quark and antiquark happen to have small relative velocities. In contrast, a point coupling gives an amplitude for meson formation independent of relative velocity. To remedy this situation, and introduce the appropriate nonlocality, consider an effective interaction with an arbitrary number of derivatives:

$$-i\sum_{n=0}^{\infty} a_n \bar{Q}(x)\left(\frac{i\overleftrightarrow{\partial}\cdot i\overleftrightarrow{\partial}}{M^2}\right)^n \gamma_5 Q(x)\phi(x)\ .$$

Here a_n are dimensionless numbers, to be determined later, and M is the quarkonium mass. Before we show that this leads to conventional formulas for decays etc., we ask what is its gauge-invariant generalization. This is too unwieldy in general. But as discussed earlier, perturbation theory holds in this model, and so we can meaningfully consider a gauge-invariant model at the one gluon level. The most general form of this must be

$$\begin{aligned}\mathcal{L}_P = &-i\sum_{n=0}^{\infty} a_n \bar{Q}\gamma_5\left(\frac{i\overleftrightarrow{D}\cdot i\overleftrightarrow{D}}{M^2}\right)^n Q\phi\\ &-\frac{ig}{2m^2}\sum_{n=0}^{\infty} b_{n+1}\bar{Q}\gamma_5\sigma_{\mu\nu}\\ &\times\left\{\left(\frac{i\overleftrightarrow{D}\cdot i\overleftrightarrow{D}}{M^2}\right)^n F^{\mu\nu}\right\}_{\rm sym} Q\phi\ ,\end{aligned} \tag{6}$$

where $\overleftrightarrow{D}^\mu$ is the usual covariant derivative,

$$\frac{i}{2}\overleftrightarrow{D}^\mu = \frac{i}{2}(\vec{\partial}^\mu - \overleftarrow{\partial}^\mu) - g\frac{\lambda_a}{2}A_a^\mu\ , \tag{7}$$

and the symmerization braces are defined by

$$\begin{aligned}\{\,\}_{\rm sym} = &(i\overleftrightarrow{D}\cdot i\overleftrightarrow{D})^n F^{\mu\nu} + (i\overleftrightarrow{D}\cdot i\overleftrightarrow{D})^{n-1}F^{\mu\nu}(i\overleftrightarrow{D}\cdot i\overleftrightarrow{D})\\ &+\cdots + F^{\mu\nu}(i\overleftrightarrow{D}\cdot i\overleftrightarrow{D})^n\ .\end{aligned} \tag{8}$$

The interaction in Eqs. (6)–(8), consisting of an infinite tower of operators grouped together by symmetry, is, in a sense, a twist expansion of the type encountered in hard processes. Here the "hard momentum" is the quarkonium mass. Equation (6) is complete at the leading twist level; other terms added on to it will be subdominant. The model leads in a straightforward manner to Feynman vertices.[1] These are discussed below.

A. *P*-quark vertex [Fig. 1(a)]

The vertex factor for pseudoscalar coupling to quarks is

$$\gamma_5 F(p^2)\ , \tag{9}$$

where

$$F(p^2) = \sum_{n=0}^{\infty} a_n\left(\frac{p^2}{m^2}\right)^n\ , \tag{10}$$

and $p^\mu = \frac{1}{2}(p_2 - p_1)^\mu$ is the relative momentum.

Do we have any intuition about $F(p^2)$? Since $(p_1 + p_2)^2 = M^2$, it follows that $4p^2 = 2p_1^2 + 2p_2^2 - M^2$ approaches zero for $\varepsilon = 2m - M$ approaching zero, i.e., the weak binding limit. It is therefore reasonable to expect that $F(p^2)$ is steeply peaked around $p^2 = 0$. These explanations are confirmed in the next section, where it will be shown that $F(p^2)$ can be expressed directly in terms of the nonrelativistic wave function of the quarks.

B. *P*-quark-electric gluon vertex [Fig. 1(b)]

The vertex factor for coupling to an E_1 gluon originates from expanding out the covariant derivatives in the first term of Eq. (6) and keeping a single gluon operator only. Taking the matrix element indicated in Fig. 1(b) and organizing the terms suitably leads to a rather nice and compact form:

$$\frac{2g}{m^2}\gamma_5\varepsilon\cdot p\mathcal{G}(p,q)\ . \tag{11}$$

The function $\mathcal{G}(p,q)$ is most easily expressed in terms of the dimensionless variables y and z:

$$\mathcal{G}(p,q) = \frac{F(y) - F(z)}{y - z}\ , \tag{12}$$

[1]Equation (6) contains derivatives and therefore $H_P \neq -L_P$. The presence of derivatives leads to additional terms upon quantization. To illustrate, suppose $\mathcal{L} \sim g\bar{\psi}\gamma^\mu\psi\partial_\mu\Phi$. Then the Hamiltonian contains a term proportional to $g^2(\psi^\dagger\psi)^2$. Quartic and higher self-couplings can be neglected at the order of accuracy of our calculations.

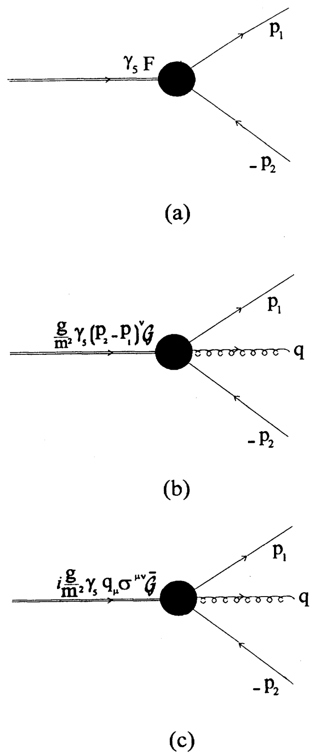

FIG. 1. Vertex factors for the coupling of pseudoscalar quarkonium to quarks and gluons. (a) Quarkonium-quarks coupling; (b) and (c) coupling to electric and magnetic gluons respectively.

where

$$y = \frac{(p + \frac{1}{2}q)^2}{m^2} \,, \qquad z = \frac{(p - \frac{1}{2}q)^2}{m^2} \,. \tag{13}$$

Note that this vertex is directly expressible in terms of $F(p^2)$ and, hence, as we shall see, in terms of nonrelativistic quark wave function. This is a simple and direct consequence of gauge invariance, and involves no new assumptions. This is not true for the other gluon vertex discussed below.

C. *P*-quark-magnetic gluon vertex [Fig. 1(c)]

This vertex follows from systematically expanding the second term in Eq. (6), keeping only one gluon, and taking the matrix element indicated in Fig. 1(c). This yields

$$i\frac{g}{m^2}\gamma_5\sigma_{\mu\nu}\varepsilon^\nu q^\mu \tilde{\mathcal{G}}(p,q) \,, \tag{14}$$

with

$$\tilde{\mathcal{G}}(p,q) = \frac{\tilde{F}(y) - \tilde{F}(z)}{y - z} \,, \tag{15}$$

and

$$\tilde{F}(p^2) = \sum_{n=1}^{\infty} b_n \left(\frac{p^2}{m^2}\right)^n \,. \tag{16}$$

$\tilde{F}(p^2)$ is in principle different from $F(p^2)$ although one can expect a similar functional dependence.

The symmetry of operators will be different in different mesons. Since we shall deal with J/ψ decays, it is useful to consider the generalization of the pseudoscalar results. All vertices in Fig. 1 are immediately applicable to the 1^{--} system by substituting $\gamma_5 \to -i\gamma^\mu$ and contracting with the meson polarization vector. In the limit of large M, the spin-spin interaction is weak and therefore $F_P(p^2) = F_V(p^2)$.

III. CONVENTIONAL LIMIT

Now that the Feynman vertices for the model have been made explicit, several calculations can be done straightforwardly. But first, to understand the physics of $F(y^2)$, consider the lowest-order diagram [Fig. 2(a)] contributing to the electric form factor. The contact term [Fig. 2(b)] involves a higher power of p^2 and is therefore neglected. Imagine that only the quark has an electric charge and the antiquark is uncharged. The amplitude, to leading order in the quark relative momentum p and for small photon momentum q^μ, is

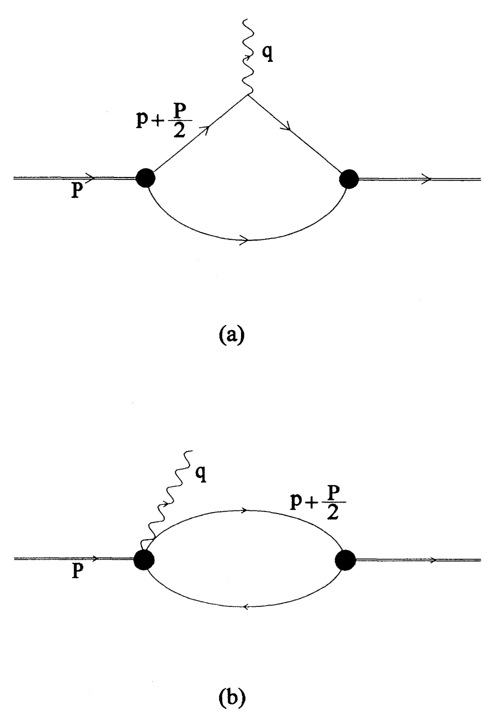

FIG. 2. Contributions from (a) direct and (b) contact terms to the form factor of $^1S_0$.

$$\mathcal{A} = 2eM^2\varepsilon^* \cdot P \int \frac{d^4p}{(2\pi)^4} \frac{F(p^2)F[(p-\frac{1}{2}q)^2]}{[(p+\frac{1}{2}P)^2 - m^2 + i\varepsilon][(p-\frac{1}{2}P)^2 - m^2 + i\varepsilon][(p+\frac{1}{2}P - q)^2 - m^2 + i\varepsilon]} \ . \tag{17}$$

The p^0 integration may be performed by keeping only the contributions of poles in the vicinity of small p^0. This yields, for $q^2 = 0$,

$$\mathcal{A} = -2ie\varepsilon^* \cdot P \int d^3p \left[\frac{F(\mathbf{p}^2)}{(2\pi)^{3/2}M^{1/2}(\varepsilon + \mathbf{p}^2/m)}\right]^2 \ . \tag{18}$$

On the other hand, if we consider a charged spinless particle scattering from an em field, this has an amplitude equal to $-2ie\varepsilon^* \cdot P$. This allows us to identify the factor in the brackets in Eq. (18) with the nonrelativistic quark wave function:

$$\begin{aligned}\frac{F(\mathbf{p}^2)}{(2\pi)^{3/2}M^{1/2}(\varepsilon + \mathbf{p}^2/m)} &= \psi(\mathbf{p}) \ , \\ &= \frac{1}{\sqrt{4\pi}}\mathcal{R}(p) \ .\end{aligned} \tag{19}$$

As a consistency check, we calculate the $\eta \to 2\gamma$ decay in the model defined by Eq. (6). In the c.m. frame both photons have large energy $M/2$, and therefore the contact term in Fig. 3(b), which is sharply damped by the form factor $\mathcal{G}$, does not contribute. Figure 3(a) and its crossed version yield, for the amplitude,

$$\mathcal{A} = 2e^2M\varepsilon^{\alpha\beta\gamma\delta}\varepsilon^*_\alpha(q_1)\varepsilon^*_\beta(q_2)q_{1\gamma}q_{2\delta}\mathcal{I} \ , \tag{20}$$

where

$$\mathcal{I} = \int \frac{d^4p}{(2\pi)^4} \frac{F(p^2)}{[(p+\frac{1}{2}P)^2 - m^2 + i\varepsilon][(p-\frac{1}{2}P)^2 - m^2 + i\varepsilon][(p+\frac{1}{2}q_2 - \frac{1}{2}q_1)^2 - m^2 + i\varepsilon]} \ . \tag{21}$$

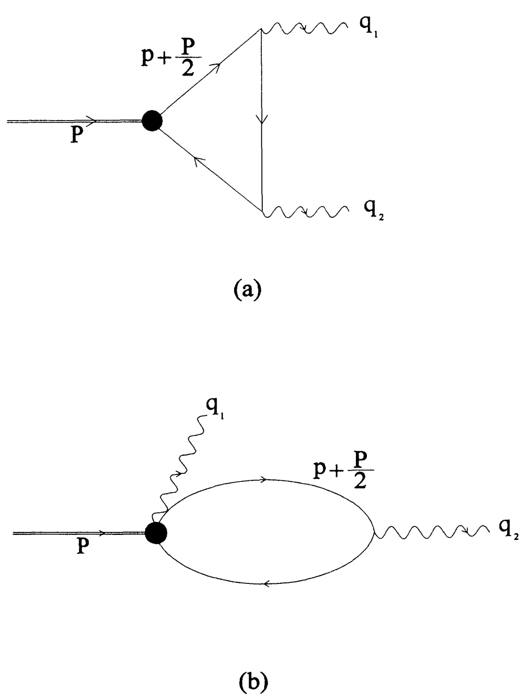

FIG. 3. Contributions from (a) direct and (b) contact terms to the $\eta \to 2\gamma$ decay rate.

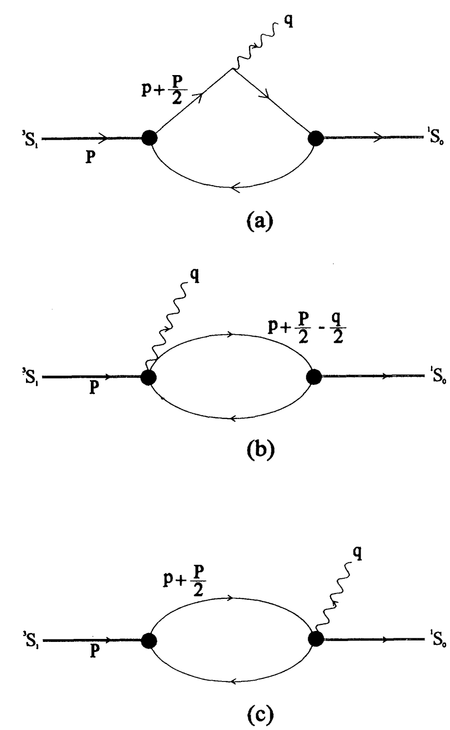

FIG. 4. Probing the gluon distribution in a $^1S_0$ meson: (a) gluon and unobserved final state hadrons X; (b) gluon emission from a quark line; and (c) gluon emission from $^1S_0$ vertex.

Performing the p^0 integration as before, using Eq. (19) to relate $F(p^2)$ to the wave function, doing the final state, phase space, integration, and summing over colors, one arrives at the standard expression for $\eta \to 2\gamma$ decays:

$$\Gamma_{\eta \to 2\gamma} = 12\alpha^2 \frac{|R(0)|^2}{M^2} . \tag{22}$$

Only $R(0)$ enters the expression, which is natural enough since the two quarks annihilate only when very close together [and a check of the model in Eq. (6)]. Gauge invariance of the quarkonium state does not play a significant role in this process.

The emission of a soft gluon or photon, as in the $M1$ transition $J/\psi \to \eta_c + \gamma$ (Fig. 4), does bring forth the issue of gauge invariance in an important way because the contact diagrams [Figs. 4(b) and 4(c)] are unsuppressed. The amplitude for the process is

$$\mathcal{A} = -4i \frac{e}{M} \varepsilon^{\alpha\beta\gamma\delta} \varepsilon^*_\alpha(q) \varepsilon_\beta(P) P_\gamma q_\delta (\mathcal{I}_{\rm dir} + \mathcal{I}_{\rm con}) . \tag{23}$$

Here $M, P^\mu, \varepsilon^\mu(P)$ refer to the J/ψ. The "direct," or conventional term, follows from [Fig. 5(a)] and its crossed version:

$$\mathcal{I}_{\rm dir} = iM^2 \int \frac{d^4p}{(2\pi)^4} \frac{F_{J/\psi}(p) F_\eta(p - \frac{1}{2}q)}{[(p + \frac{1}{2}P)^2 - m^2 + i\varepsilon][(p - \frac{1}{2}P)^2 - m^2 + i\varepsilon][(p + \frac{1}{2}P - q)^2 - m^2 + i\varepsilon]} . \tag{24}$$

Performing the p^0 integration and keeping only the contributions from the poles in the lower half plane near $p^0 \approx 0$ yields

$$\mathcal{I}_{\rm dir} = M \int \frac{d^3p}{(2\pi)^3} \frac{F_{J/\psi}(p) F_\eta(p - \frac{1}{2}q)}{(\varepsilon + \mathbf{p}^2/m)^2}$$
$$= \int dr r^2 \exp\left(\frac{i}{2}\mathbf{q}\cdot\mathbf{r}\right) R_{J/\psi}(r) R_\eta(r) . \tag{25}$$

In the above, we have kept only the leading-order term in the photon energy q^0. In the dipole approximation, the exponential factor is unity and if the two mesons had identical wave functions then one would simply have $\mathcal{I}_{\rm dir} = 1$. Because of the hyperfine splitting this deviates from unity, and a typical (model-dependent) value [5] is $\mathcal{I}_{\rm dir} = 0.987$. Another set of model parameters used by the same authors yields 0.984, 0.920. From the amplitude, Eq. (23), the decay width is readily seen to be

$$\Gamma_{\rm dir}(^3S_1 \to \gamma + ^1S_0) = \frac{4}{3}\alpha e^2 \left(\frac{q}{m}\right)^2 q |\mathcal{I}|^2$$
$$\approx 2.41 \text{ keV} . \tag{26}$$

This differs substantially from the measured width, 1.11 ± 0.35 keV. This is a well-known problem with the usual charmonium model, and a wide range of explanations exist for the factor of 2–3 discrepancy. These include relativistic corrections, missing effects, anomalous quark magnetic moment, etc. References to these may be found in a recent review by Schuler [6].

The contact terms in Figs. 4(b) and 4(c), which are required by gauge invariance, may also be calculated quite straightforwardly:

$$\mathcal{I}_{\rm con} = 4i \int \frac{d^4p}{(2\pi)^4} \frac{\tilde{\mathcal{G}}(p,q) F(p)}{[(p - \frac{1}{2}P + \frac{1}{2}q)^2 - m^2 + i\varepsilon][(p + \frac{1}{2}P - \frac{1}{2}q)^2 - m^2 + i\varepsilon]} . \tag{27}$$

Since this is a correction term, it is adequate to take $F_{J/\psi} = F_\eta = F$ and $\mathcal{G}_{J/\psi} = \mathcal{G}_\eta = \mathcal{G}$. The p^0 integration gives, keeping only the contribution of nearly coincident poles near the origin,

$$\mathcal{I}_{\rm con} = -\frac{4}{M^2} \int \frac{d^3p}{(2\pi)^3} \frac{\tilde{\mathcal{G}}(p,q) F(p)}{\varepsilon + p^2/m}$$
$$= -\frac{2\sqrt{2}}{\pi M^{3/2}} \int dp p^2 \tilde{\mathcal{G}}(p,q) R(p) . \tag{28}$$

Since $|\mathbf{q}| \ll |\mathbf{p}|$, it is adequate to replace $\mathcal{G}$ in Eq. (15) by

$$\tilde{\mathcal{G}}(p) = -\frac{m^2}{2p} \frac{d\tilde{F}}{dp} . \tag{29}$$

At this point one needs to confront the following issue: what is $\tilde{F}$ equal to? We have seen that F is related to the electric charge distribution and is expressible in terms of the nonrelativistic (NR) wave function [Eq. (19)]. It is possible to show that $\tilde{F}$ is related to the magnetic response of the system and can, in principle, also be found from a NR quark model (NRQM) calculation. But this is not immediately useful as these calculations have been done only for static quantities. Instead, we make the physically plausible assumption that $\tilde{F}(p) = \xi F(p)$ where ξ is a scale factor. Substituting this into Eqs. (29) and (28) yields

$$\mathcal{I}_{\rm con} = -\frac{1}{2}\xi M \int_0^\infty dp(\varepsilon + p^2/m) R(p) \frac{d}{dp} p R(p)$$
$$= \frac{1}{2}\xi \left(1 - \frac{M\varepsilon}{2} \left\langle \frac{1}{p^2} \right\rangle \right) . \tag{30}$$

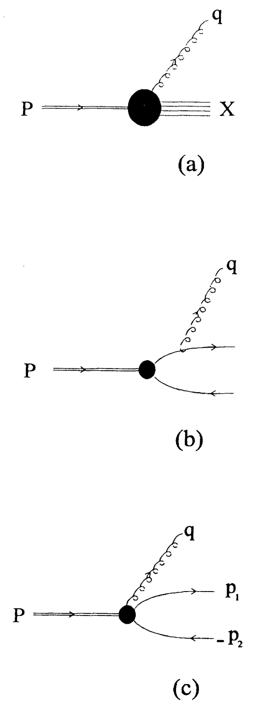

FIG. 5. Contributions to $^3S_1 \to \gamma +^1 S_0$ decay rate from: (a) direct diagram; (b) contact diagram with γ emanating from $^3S_1$ vertex; and (c) contact diagram with γ emanating from $^1S_0$ vertex.

Any given quarkonium model allows for the calculation of ε and $\langle 1/p^2\rangle$. For definiteness, assume a Gaussian wave function of the type $\exp(-p^2/2\beta^2)$, which yields $\langle 1/p^2\rangle = 2/\beta^2$. Fitting to the η_c decay rate gives $\beta^2 \approx 0.413$ GeV^2. With $m_c = 1.65$ GeV, a commonly used value for the charm quark mass, it follows that $\mathcal{I}_{\rm con} = -0.65\xi$. Now suppose that the entire (large) discrepancy between the measured decay width and the conventionally calculated value can be attributed to the magnetization term which, as we have argued, symmetry requirements force us to include in the Lagrangian, Eq. (6). The total decay rate is

$$\Gamma(^3S_1 \to \gamma +^1 S_0) = \frac{4}{3}\alpha e_Q^2 \left(\frac{q}{m}\right)^2 q(\mathcal{I}_{\rm dir} + \mathcal{I}_{\rm con})^2$$
$$= 1.11 \pm 0.35 \ (\text{for } J/\psi \to \gamma + \eta_c) \ . \quad (31)$$

This suggests that ξ is a number around 0.33–0.66. With the magnetization term thus determined, one can make physical predictions for quarkonium processes involving soft photons or gluons. An application to gluon fragmentation into heavy quarkonium will be described in the section after next.

IV. GLUONIC DISTRIBUTION

In the present model it is possible to calculate the light-cone distribution of low momentum gluons in a weakly bound heavy quarkonium system. It will be shown that requiring gauge invariance of the hadronic state implies the existence of a term which goes as $x^{-1}\ln x$, which is more singular than the x^{-1} dependence calculated by Brodsky and Schmidt [7] using simple perturbative arguments for positronium. An explicit expression for the coefficient of the $x^{-1}\ln x$ term can be provided in terms of the nonrelativistic wave function. Although calculable, we shall not worry about the x^{-1} terms as this is anyway a theoretical exercise—stable quarkonium targets unfortunately do not exist, and so gluonic distribution is not directly measurable.

The starting point [8] is the formula for the gluon momentum distribution inside a spinless hadronic target, written as a correlation of operators on the light cone:

$$G(x) = x\int \frac{d\lambda}{2\pi} e^{i\lambda x}\langle P|A^i(0)A^i(\lambda n)|P\rangle \ . \quad (32)$$

Here k^μ and n^μ are two null vectors $k^2 = n^2 = 0$ with $k\cdot n = 1$, $k^- = n^+ = 0$, and $\mathcal{P}^\mu = k^\mu + \frac{1}{2}M^2 n^\mu$.

Only the transverse components of the gluon field are involved. Inserting a complete set of states between the two operators, and limiting the outgoing X to two quarks only, yields

$$G(x) = x\int [dp_1][dp_2]\delta(x - q\cdot n)|\langle p_1p_2|A^i|P\rangle|^2 \ . \quad (33)$$

A summation on physical gluon polariations ($i = 1,2$), as well as color indices ($a = 1,8$), is implicit. The measure $[dp]$ is

$$[dp] = \frac{dp^+ d^2p_\perp}{2p^+(2\pi)^3} \ . \quad (34)$$

We now concentrate upon calculating the matrix element in Eq. (33). If we limit our interest to the terms most singular in x, it turns out that the term corresponding to radiation from a quark line [Fig. 4(b)] can be ignored.[2] The emission of a magnetic gluon from the vertex [Fig. 4(c)] is also subdominant—this follows because the amplitude, Eq. (14), vanishes as $q \to 0$. The dominant contribution comes from the emission of an electric gluon. Keeping just this term, we have from Eq. (11), that

$$\langle p_1p_2|A^\mu|P\rangle = 2\frac{g}{m^2}\bar{u}(p_1)\gamma_5 v(p_2)\frac{iD^{\mu\nu}}{q^2}p_\nu \mathcal{G} \ . \quad (35)$$

The light-cone propagator is used in the above:

$$D^{\mu\nu} = -g^{\mu\nu} + \frac{q^\mu n^\nu + q^\nu n^\mu}{q\cdot n} \ . \quad (36)$$

Keeping only the most singular term at small x, as well as the lowest-order term in the quark relative momentum $p^\mu = \frac{1}{2}(p_1^\mu - p_2^\mu)$, and performing the polarization sum, the squared matrix element is calculated to be

[2] This is the diagram which yields the x^{-1} behavior in the work of Brodsky and Schmidt [7].

$$\sum |\langle p_1 p_2 | A^i | P\rangle|^2 = \frac{32g^2}{m^2x^2}\frac{q_\perp^2 (p\cdot n)^2}{q^4}\mathcal{G}^2 \ . \qquad (37)$$

Again, for small x,

$$q^2 = -x^2M^2 - q_\perp^2 \ , \qquad (38)$$

$$[dp_1][dp_2]\delta(x - q\cdot n) = \frac{dp^+ d^2p_\perp d^2K_\perp}{P^+(2\pi)^6} \ , \qquad (39)$$

where $K_\perp = p_{1\perp} + p_{2\perp}$. This yields

$$G(x) = \frac{16g^2}{m^4}\frac{1}{xP^+}\int \frac{dp^+ d^2p_\perp d^2K_\perp}{(2\pi)^6}\frac{K_\perp^2 p^{+2}}{(x^2M^2 + K_\perp^2)^2}\mathcal{G}^2 \ . \qquad (40)$$

From Eqs. (12) and (13) $\mathcal{G}$ is related to the wave function Eq. (19) and its derivative through

$$\mathcal{G} = \frac{dF}{dt} \ , \qquad (41)$$

where t is a dimensionless variable:

$$\begin{aligned} t &= \frac{(p^2 + \frac{1}{4}q^2)}{m^2} \\ &= -\left(x^2 + \frac{K_\perp^2}{4m^2} + \frac{p_\perp^2}{m^2} + \frac{2p^{+2}}{m^2}\right) \\ &= -x^2 - \eta^2 \ . \end{aligned} \qquad (42)$$

Performing the angular integration yields, for the gluon distribution,

$$G(x) = -A\frac{\ln x}{x} + O\left(\frac{1}{x}\right) \ , \qquad (43)$$

where A is a positive constant determined by the nonrelativistic quark wave function,

$$A = \frac{2\alpha}{3\pi^3}\int_0^\infty d\eta\eta^4\mathcal{G}^2(\eta) \ . \qquad (44)$$

This is the main result of this section. It shows that demanding gauge invariance of the hadronic state has a profound effect upon the distribution of low momentum gluons.

V. GLUON FRAGMENTATION

Our final application of the model developed in this paper is to calculate the rate of fragmentation of gluons into quarkonia. Gluon fragmentation refers to the process of converting highly virtual gluons into hadronic physical states. The calculation is done in two parts. First, the fragmentation function is calculated at the scale of the heavy quark mass, and, second, it is evolved perturbatively from low to high virtualities. If one assumes that perturbative QCD is valid even at the scale of the charm quark mass, then a first principles calculation of fragmentation becomes possible. Recently Braaten and Yuan [9] have performed such a calculation for gluons fragmenting into $^1S_0$ and $^3S_1$ heavy mesons.

In the first part of this section we show that the result of the calculation of Braaten and Yuan [9] can be exactly replicated by considering the direct diagrams [Fig. 6(a)] implied by the model. The only difference is that our calculation can be performed entirely in field theoretical language, which is perhaps an advantage. In the second part, we show that long wavelength magnetic gluons emitted by the contact diagram [Fig. 6(b)] augment the previous contribution. We remind the reader that, in the present model, "long wavelength" nevertheless means a wavelength sufficiently small for perturbative QCD to be valid: as discussed earlier there is a hierarchy of scales, $\Lambda_{\rm QCD} \ll q \ll p \ll M$.

The starting point of the calculation is the expression for the unpolarized gluon fragmentation function into a specific quarkonium state with momentum P^μ. The reference frame is chosen to be the rest frame of the hadron:

$$D(z) = \frac{1}{2}\int \frac{d\lambda}{2\pi} e^{-i\lambda/z}\langle 0|A^i(0)|PX\rangle\langle PX|A^i(\lambda n)|0\rangle \ . \qquad (45)$$

The notation here is identical to that in the previous section, i.e., $k^2 = n^2 = 0$, etc. The expression for $D(z)$ follows from duplicating the analysis of Jaffe and Ji [10] and replacing quark operators with gluon operators. Since $D(z)$ involves only the "good" components of the gluon field, it is a twist two quantity. A sum on the unobserved states X is implied. To see more clearly the physical meaning of Eq. (45), put $|PX\rangle = C^\dagger(P)|X\rangle$, where $C^\dagger(P)$ creates a meson of a given type. Using completeness of the states $|X\rangle$ gives

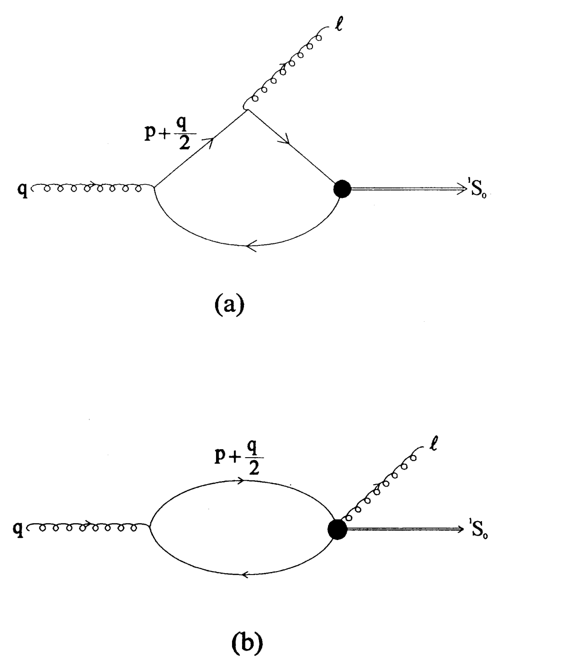

FIG. 6. Gluon fragmenting into $\eta(^1S_0)$. (a) Direct diagram and (b) contact diagram.

$$D(z) = \int [dq]\delta\left(\frac{p^+}{P^+} - \frac{1}{z}\right) \frac{\langle q|C^\dagger(P)C(P)|q\rangle}{\langle q|q\rangle} . \quad (46)$$

This makes apparent that $D(z)$ is essentially the probability of finding a specific hadron with + component of momentum equal to zq^+. The transverse momentum $q_\perp$ of the incoming gluon is integrated over.

Now consider the production of a $^1S_0$ state from the process in Fig. 6(a). After calculating traces, the amplitude for the process is

$$\langle PX|A^\mu|0\rangle = \frac{iD^\mu_\nu}{q^2}\mathcal{A}^\nu_{\rm dir} , \quad (47)$$

$$\mathcal{A}^\nu_{\rm dir} = -8g^2 m\varepsilon^{\nu\alpha\beta\gamma}P_\gamma\varepsilon^*_\beta l_\alpha \mathcal{I}_{\rm dir} , \quad (48)$$

$$\mathcal{I}_{\rm dir} = \int \frac{d^4p}{(2\pi)^4} \frac{F(p-\frac{1}{2}l)}{[(p+\frac{1}{2}q)^2 - m^2 + i\varepsilon][(p-\frac{1}{2}q)^2 - m^2 + i\varepsilon][(p+\frac{1}{2}q - l)^2 - m^2 + i\varepsilon]} . \quad (49)$$

The crossed diagram doubles the above value of $\mathcal{I}$. Since the form factor, which is essentially the wave function, restricts the relative quark momentum to small values, it is apparent from the denominators in Eq. (49) that the dominant contribution to the integral comes from the region around $q^2 \approx 4m^2$. Performing the p^0 integral as in the previous applications, and using Eq. (19), yields, for the direct amplitude,

$$\langle PX|A^\mu|0\rangle_{\rm dir} = \frac{8g^2}{M^{1/2}} \frac{\psi(0)}{s(s-M^2)} D^{\mu\nu}\varepsilon_{\nu\alpha\beta\gamma} l^\alpha \varepsilon^{*\beta} P^\gamma , \quad (50)$$

where $s = q^2$ is the mass of the fragmenting gluon, and we have set $M^2 \approx 4m^2$. The sum over unobserved states in Eq. (45), which amounts to an integration over the gluon momentum l, leads to

$$D(z) = \frac{1}{6\pi^2} \frac{g^4\psi^2(0)}{M} \int_{M^2/z}^\infty \frac{ds}{s^2} \frac{[(1-2z+2z^2)s^2 - 8sm^2z + 16m^4]}{(s-M^2)^2} . \quad (51)$$

The lower limit of integration follows from setting the minimum value of $q_\perp^2 = l_\perp^2 = 0$ in

$$s = \frac{M^2}{z} + \frac{z}{1-z}q_\perp^2 . \quad (52)$$

A color factor of 1/12 has been included in Eq. (51). Performing the integration yields

$$D(z) = \frac{\alpha_s^2}{3\pi} \frac{|R(0)|^2}{M^3}[3z - 2z^2 + 2(1-z)\ln(1-z)] . \quad (53)$$

This is precisely the result of Braaten and Yuan [9], i.e., Eq. (8) of their paper.

The contact diagram of Fig. 6(a) is calculated similarly, and it has the Lorentz structure given in Eq. (48) with

$$\mathcal{I}_{\rm con} = \frac{2}{M^2} \int \frac{d^4p}{(2\pi)^4} \frac{\tilde{\mathcal{G}}(p,l)}{[(p+\frac{1}{2}q)^2 - m^2 + i\varepsilon][(p-\frac{1}{2}q)^2 - m^2 + i\varepsilon]} . \quad (54)$$

Only the vertex equations (14)–(16) are involved; electric gluons do not contribute here. Since the magnetic form factor $\tilde{\mathcal{G}}$ restricts p to small values, from the two denominators in the above integral it is evident that the major contribution comes from small values of the outgoing gluon momentum l. Performing the p^0 integration gives

$$\langle PX|A^\mu|0\rangle_{\rm con} = -\frac{8g^2}{M^3 s|\boldsymbol{l}|} D^{\mu\nu}\varepsilon_{\nu\alpha\beta\gamma} l^\alpha \varepsilon^{*\beta} P^\gamma \int \frac{d^3p}{(2\pi)^3}\tilde{\mathcal{G}} . \quad (55)$$

Since we have chosen our reference frame as the rest-frame of the produced meson, it follows that

$$|\boldsymbol{l}| = (s - M^2)/2M . \quad (56)$$

Using Eq. (29) yields

$$\int \frac{d^3p}{(2\pi)^3}\tilde{\mathcal{G}}(p) = -\frac{m^2}{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{p}\frac{d\tilde{F}}{dp} = \frac{1}{4}M^3\left(\frac{\tilde{\psi}(0)}{M^{3/2}} + \frac{\varepsilon}{2M}M^{1/2} \times \int_0^\infty dr r\tilde{\psi}(r)\right) . \quad (57)$$

Thus the total amplitude is

$$\langle PX|A^{\mu}|0\rangle_{\rm dir} + \langle PX|A^{\mu}|0\rangle_{\rm con} = \frac{8e^2}{M^{1/2}}\frac{1}{s(s-M^2)}D^{\mu\nu}\varepsilon_{\nu\alpha\beta\gamma}l^{\alpha}\varepsilon^{*\beta}P^{\gamma}\left((\psi(0) - \tfrac{1}{2}\chi(0)\right) , \quad (58)$$

where

$$\chi(0) = \tilde{\psi}(0) + \frac{\varepsilon M}{2}\int_0^{\infty} dr r\tilde{\psi}(r) . \quad (59)$$

Using Eqs. (58) and (59) to calculate the fragmentation function yields

$$D(z) = \frac{\alpha_s^2}{3\pi M^3}\eta(z)\left(R(0) - \tfrac{1}{2}S(0)\right)^2 , \quad (60)$$

where

$$\eta(z) = 3z - 2z^2 + 2(1-z)\log(1-z) \quad (61)$$

and

$$S(0) = \sqrt{4\pi}\chi(0) . \quad (62)$$

The first term in Eq. (60) is the direct term and is the same as Eq. (53), while the second is the contact term. To estimate numerically $D(z)$, we shall make the same assumption as in Sec. III, i.e., that $\tilde{F}(p) = \xi F(p)$ and use the same Gaussian wave function. $D(z)$ can then be expressed as

$$D(z) = \frac{\alpha_s^2}{3\pi M^3}\eta(z)R^2(0)(1-a)^2 , \quad (63)$$

where a varies from 0.35 to 0.71 as ξ goes from 0.33 to 0.66, the range estimated in Sec. III. It is clear from Eq. (63) that except for a scale factor $(1-a)^2$ the fragmentation function at the initial scale, as well as after evolution, is identical to that calculated by Braaten and Yuan [9]. The scale factor, however, causes a substantial decrease in the magnitude of $D(z)$,

VI. SUMMARY

We have presented in this paper a low-energy, effective, gauge-invariant theory wherein the fundamental degrees of freedom are quarkonia, quarks, and gluons. The interaction has the form of a twist expansion familiar from hard processes and consists of towers of operators grouped together according to their symmetry. The quarkonium mass plays the role of the "hard momentum." The arbitrary number of derivatives in the theory serve to bring in the appropriate amount of nonlocality or, equivalently, a form factor in momentum space which embodies the extended structure of the meson. This form factor, which is the basic input into the model, was shown to be directly related to the nonrelativistic quark wave function, a quantity calculable in any given potential model.

An important consequence of gauge invariance is the emergence of Feynman vertices representing the direct gluon-quark-antiquark-meson interaction. These vertices have a substantial effect upon certain heavy-meson phenomena. For example, the radiative M_1 transition, $J/\psi \to \eta_c + \gamma$ receives an additional contribution from one such vertex. This could help explain why the usual decay calculations invariably overestimate the decay rate by a factor of 2–3. As another example, we have calculated the light-cone momentum distribution of gluons in heavy quarkonia. Although these distributions are probably of no practical interest, nevertheless the present model does have some interesting theoretical consequences. We find that the $Q\bar{Q}G^1S_0$ vertex, which contributes to $G(x)$, not only gives the x^{-1} behavior but also has a $x^{-1}\ln x$ part which is more singular at small x.

As the final application of our model we considered the fragmentation of gluons into $^1S_0$ mesons. If we ignore the contact (gauge) diagrams, then the results of Braaten and Yuan [9] are exactly recovered. But including these diagrams leads to a downward rescaling of the their results by an amount which could be substantial. A proper calculation depends upon knowledge of the magnetic form factor, called $\tilde{F}(p^2)$ here, which in principle could be determined from a quark model calculation that includes states of arbitrary excitation. Finally, we remark that the model discussed in this paper is extendable to p states as well. This would be interesting because the centrifugal barrier keeps the quarks relatively further apart, and thus makes the issue of nongauge invariance of the meson state more acute.

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