Spin Structure of the Nucleon in the Asymptotic Limit

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In analogy with the Altarelli-Parisi equation for quark and gluon helicity contributions to the nucleon spin, we derive an evolution equation for quark and gluon orbital angular momenta. Solution of the combined equations yields asymptotic fractions of the nucleon spin carried by quarks and gluons: $\Delta\Sigma/2 + L_q = 3n_f/(16 + 3n_f)$ and $\Delta g + L_g = 16/(16 + 3n_f)$, respectively, with n_f the number of active quark flavors. These are identical to the well-known asymptotic partitions of the nucleon momentum between quark and gluon contributions. The axial-anomaly contribution to the quark helicity is canceled with a similar contribution to the quark orbital angular momentum, making the total quark contribution to the nucleon spin anomaly free.

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What is the composition of the nucleon spin in terms of its quark and gluon constituents? In the last few years, the answer has become the holy grail for the nuclear and particle spin community. The motivation is quite clear: given quantum chromodynamics (QCD) is difficult to solve, such information gives us valuable insights into the nonperturbative wave function of the nucleon. A satisfactory understanding of the spin structure would be to know, for instance, how much spin of the nucleon is carried, respectively, by the quark and gluon helicities and orbital angular momenta, in the same way as one now understands how the mass of the nucleon is partitioned among contributions from the quark and gluon kinetic energies, quark masses, and the trace anomaly [1].

The theoretical basis for separating the spin of the nucleon into different contributions begins with the QCD expression for the generators of Lorentz transformations [2],

$$J^{\mu\nu} = \int d^3x\, M^{0\mu\nu}, \tag{1}$$

where $M^{\alpha\mu\nu}$ is the angular momentum density,

$$\begin{aligned} M^{\alpha\mu\nu} = {} & i\overline{\psi}\gamma^\alpha(x^\mu\partial^\nu - x^\nu\partial^\mu)\psi + \frac{i}{4}\overline{\psi}\gamma^\alpha \\ & \times [\gamma^\mu, \gamma^\nu]\psi - F^{\alpha\sigma}(x^\mu\partial^\nu - x^\nu\partial^\mu)A_\sigma \\ & - F^{\alpha\mu}A^\nu + F^{\alpha\nu}A^\mu, \end{aligned} \tag{2}$$

where color indices are implicit. According to definition, a nucleon moving in the z direction with momentum P^μ and helicity 1/2 satisfies

$$J^{12}|P+\rangle = \tfrac{1}{2}|P+\rangle. \tag{3}$$

Thus one can write down a spin sum rule

$$\begin{aligned} \tfrac{1}{2} &= \langle P+|J^{12}|P+\rangle/\langle P+|P+\rangle \\ &= \tfrac{1}{2}\Delta\Sigma + \Delta g + L_q + L_g, \end{aligned} \tag{4}$$

where the matrix elements are defined as ($\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$)

$$\begin{aligned} \Delta\Sigma &= \langle P+|\hat{S}_{3q}|P+\rangle \\ &= \langle P+|\int d^3x\,\overline{\psi}\gamma^3\gamma_5\psi|P+\rangle, \\ \Delta g &= \langle P+|\hat{S}_{3g}|P+\rangle \\ &= \langle P+|\int d^3x\,(E^1A^2 - E^2A^1)|P+\rangle, \\ L_q &= \langle P+|\hat{L}_{3q}|P+\rangle \\ &= \langle P+|\int d^3x\,\overline{\psi}\gamma^0(x^1\partial^2 - x^2\partial^1)\psi|P+\rangle, \\ L_g &= \langle P+|\hat{L}_{3g}|P+\rangle \\ &= \langle P+|\int d^3x\,E^i(x^2\partial^1 - x^1\partial^2)A^i|P+\rangle, \end{aligned} \tag{5}$$

where for simplicity we have neglected the normalization of the state. It is clear from the above that $\Delta\Sigma$ and Δg are the quark and gluon helicity contributions to the nucleon spin, and L_q and L_g are the quark and gluon orbital angular momentum contributions. Apart from $\hat{S}_{3q}$, the other three operators $\hat{S}_{3g}$, $\hat{L}_{3q}$, and $\hat{L}_{3g}$ are not manifestly gauge invariant, and thus a decomposition of the nucleon spin

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is in general gauge dependent. Furthermore, the matrix elements depend on the choice of Lorentz frame. Only in light-front coordinates and light-front gauge [3] Δg is the gluon helicity measured in high-energy scattering processes. We henceforth work in these coordinates and gauge [4] [the index 0 in Eq. (1) is now replaced by $+$].

The individual operators in J^{12} are not conserved charges, and hence their matrix elements are generally divergent. They can be renormalized in a scheme and the renormalization introduces a scale dependence. The scale dependence of $\Delta\Sigma$ and Δg obeys the well-known Altarelli-Parisi (AP) equation, which in the leading-log approximation is [5]

$$\frac{d}{dt}\begin{pmatrix}\Delta\Sigma\\ \Delta g\end{pmatrix} = \frac{\alpha_s(t)}{2\pi}\begin{pmatrix}0 & 0\\ \frac{3}{2}C_F & \frac{\beta_0}{2}\end{pmatrix}\begin{pmatrix}\Delta\Sigma\\ \Delta g\end{pmatrix}, \tag{6}$$

where $t = \ln Q^2/\Lambda_{\rm QCD}^2$, $C_F = 4/3$, and $\beta_0 = 11 - 2n_f/3$ with n_f the number of quark flavors. The square matrix on the right-hand side is called the splitting matrix. Here we derive an equation for the leading-log evolution of the quark and gluon orbital angular momenta. The result is

$$\frac{d}{dt}\begin{pmatrix}L_q\\ L_g\end{pmatrix} = \frac{\alpha_s(t)}{2\pi}\begin{pmatrix}-\frac{4}{3}C_F & \frac{n_f}{3}\\ \frac{4}{3}C_F & -\frac{n_f}{3}\end{pmatrix}\begin{pmatrix}L_q\\ L_g\end{pmatrix} + \frac{\alpha_s(t)}{2\pi}\begin{pmatrix}-\frac{2}{3}C_F & \frac{n_f}{3}\\ -\frac{5}{6}C_F & -\frac{11}{2}\end{pmatrix}\begin{pmatrix}\Delta\Sigma\\ \Delta g\end{pmatrix}. \tag{7}$$

We call the first term on the right-hand side the homogeneous term and the second the inhomogeneous term (with corresponding splitting matrices). The inhomogeneous term was first studied by Ratcliffe [6], from which he concluded that orbital angular momentum plays an essential role in the nucleon spin. At the operator level, this means that the orbital angular momentum operators contain leading-twist contributions. However, our result for the inhomogeneous part of the $L_q - L_g$ evolution disagrees with Ratcliffe's. Furthermore, the homogenous term we have is new.

In the remainder of the paper, we sketch our derivation of Eq. (7) and find its general solution. At the end of the paper, we go beyond the leading-log approximation and discuss the "anomaly cancellation" in the angular momentum operator J^{12}. To keep the physics clear, we will follow as closely as possible the original language of Altarelli and Parisi in light-front coordinates although the whole discussion can be made consistently in the language of operator mixing. When necessary, we supplement our discussion with the matrix elements of angular momentum operators in composite parton states.

To begin, we review the standard derivation of the AP equation. Consider a parent quark with momentum $p^\mu = (p^- = 0, p^+, p_\perp = 0)$ and helicity $+1/2$, splitting into a daughter gluon of momentum $k^\mu = (k^-, xp^+, k_\perp)$ and a daughter quark with momentum $(p-k)^\mu$. Only 3-momentum is conserved during the splitting. The total probability for the splitting is $\int_0^1 dx\,[1 + (1-x)^2]/x$. (A multiplicative factor $\alpha_s \ln Q^2/\mu^2$ is implied when we talk about probability. μ^2 here is an infrared cutoff which defines the nature of the parent quark. It must be large enough so that perturbative QCD is valid. Q^2 is an ultraviolet transverse momentum cutoff which defines the scale of daughter partons.) Since the quark helicity is conserved at the leading-log order, we therefore have the item 0 in the upper-left corner of the AP splitting matrix. The helicity of the daughter gluon can be either $+1$ or -1. The probabilities for both cases are $\int_0^1 dx/x$ and $\int_0^1 dx(1-x)^2/x$, respectively. The gluon helicity produced in the splitting is just $\int_0^1 dx\,[1-(1-x)^2]/x = (3/2)C_F$, which is the element in the lower-left corner of the AP splitting matrix.

A further consideration of the above process leads to an inhomogeneous source for the orbital angular momenta. Since the total angular momentum in the z direction is conserved in the splitting, orbital angular momentum has to be produced to cancel the helicity of the daughter gluon. This means that total orbital angular momentum carried by the daughter quark and gluon is $-(3/2)C_F$. QCD determines partition of this between the quark and gluon according to the matrix elements,

$$\langle p+|\hat{L}_{3q}|p+\rangle, \qquad \langle p+|\hat{L}_{3g}|p+\rangle, \tag{8}$$

where $|p+\rangle$ is the quark-gluon state produced in the splitting. To calculate them, we start with less singular off-forward matrix elements and consider their forward limit [2]. We find

$$\begin{aligned}\langle p+|\hat{L}_{3q}|p+\rangle &= -C_F\mathcal{N}\int_0^1 dx\,x[1/x - (1-x)^2/x]\\ &= -\tfrac{2}{3}C_F\mathcal{N},\\ \langle p+|\hat{L}_{3g}|p+\rangle &= -C_F\mathcal{N}\int_0^1 dx\,(1-x)\left(\frac{1}{x} - \frac{(1-x)^2}{x}\right)\\ &= -\tfrac{5}{6}C_F\mathcal{N},\end{aligned} \tag{9}$$

where $\mathcal{N} = 2p^+(2\pi)^3\delta^3(0)(\alpha_s/2\pi)\ln Q^2/\mu^2$. The coefficients in front of $\mathcal{N}$ give the first column of the inhomogeneous splitting matrix in Eq. (7). Interestingly, the sharing of the orbital angular momentum is done according to the x and $1-x$ moments of the polarized gluon density in the parent quark.

A similar discussion leads to the second columns of the AP splitting matrix and the inhomogeneous splitting matrix in Eq. (7). Let us remark here the physical origin for the well-known result that the gluon helicity increases logarithmically in the asymptotic limit [7]. When a gluon splits, there are four possible final states: (1) a quark with helicity $1/2$ and an antiquark with helicity $-1/2$; (2) a quark with helicity $-1/2$ and an antiquark with helicity $1/2$; (3) a gluon with helicity $+1$ and another with helicity -1; and (4) two gluons with helicity $+1$. In the first two processes, there is a loss of the gluon helicity with probability $(n_f/2)\int_0^1 dx\,[x^2+(1-x)^2]$. In the third process, there is also a loss of the gluon helicity, with probability $\int_0^1 dx\,[x^3/(1-x)+(1-x)^3/x]$. In the last process, there is a gain of the gluon helicity with probability $\int_0^1 dx/x(1-x)$. When summed, the gluon helicity has a net gain with probability $11/2 - n_f/3 = \beta_0/2$ in the splitting. Thus at increasingly smaller distance scales, gluons split consecutively and the helicity builds up logarithmically.

Now we turn to the homogenous term in the orbital angular momentum evolution equation. To calculate the splitting matrix, we consider the matrix elements of $\hat{L}_{3q}$ and $\hat{L}_{3g}$ in a parton state with nonvanishing transverse momentum. In a bare parton state, we have

$$\langle p'+|\hat{L}_{3q}|p+\rangle = 2p^+(2\pi)^3\left(-ip_1'\frac{\partial}{\partial p_2'} + ip_2'\frac{\partial}{\partial p_1'}\right) \times \delta^3(p'-p)\,. \tag{10}$$

The derivative on the δ function means that when the distribution is convoluted with a test function, the derivative will be taken of the test function. When calculating the matrix element in the composite parton states from the parton splitting, we have

$$\langle p'+|\hat{L}_{3q}|p+\rangle = 2p^+(2\pi)^3\phi(p',p) \times\left(-ip_1'\frac{\partial}{\partial p_2'} + ip_2'\frac{\partial}{\partial p_1'}\right)\delta^3(p'-p)\,. \tag{11}$$

When the distribution is convoluted with a test function, the derivative will be taken of $\phi(p',p)$ and then of the test function. The first term represents generation of the orbital angular momentum from the parton helicity discussed previously. The second term has the same structure as the basic matrix element in Eq. (10) and represents the self-generation of orbital angular momentum in the splitting.

According to the above recipe, we calculate the gluon orbital angular momentum generated from the quark orbital angular momentum,

$$\begin{aligned}\phi(p,p) &= \frac{\alpha_s}{2\pi}\ln\frac{Q^2}{\mu^2}C_F\int_0^1 x\,\frac{1+(1-x)^2}{x}\,dx\\ &= \frac{\alpha_s}{2\pi}\ln\frac{Q^2}{\mu^2}C_F\,\frac{4}{3}\,.\end{aligned} \tag{12}$$

Interestingly, this is the same as the fraction of the quark momentum carried by the daughter gluon in the infinite momentum frame. The net loss of the quark orbital angular momentum in the splitting must be the negative of the above. These numbers form the first column of the homogeneous splitting matrix. A similar calculation yields the second column.

The solution of the evolution equation can be obtained straightforwardly. First, let us write down the well-known result for the quark and gluon helicities [6,7],

$$\begin{aligned}\Delta\Sigma(t) &= \text{const},\\ \Delta g(t) &= -\frac{4\Delta\Sigma}{\beta_0} + \frac{t}{t_0}\left(\Delta g_0 + \frac{4\Delta\Sigma}{\beta_0}\right).\end{aligned} \tag{13}$$

The second equation exhibits the famous behavior of the gluon helicity: increasing like $\ln Q^2$ as $Q^2\to\infty$. The coefficient of the term depends on the special combination of initial quark and gluon helicities, which is likely positive at low-momentum scales according to the recent experimental data on $\Delta\Sigma$ [8] and theoretical estimates for Δg [9]. The solution for the orbital momenta is

$$\begin{aligned}L_q(t) &= -\frac{1}{2}\Delta\Sigma + \frac{1}{2}\frac{3n_f}{16+3n_f}\\ &\quad + (t/t_0)^{-2(16+3n_f)/9\beta_0}\\ &\quad \times\left(L_q(0) + \frac{1}{2}\Delta\Sigma - \frac{1}{2}\frac{3n_f}{16+3n_f}\right),\\ L_g(t) &= -\Delta g(t) + \frac{1}{2}\frac{16}{16+3n_f}\\ &\quad + (t/t_0)^{-2(16+3n_f)/9\beta_0}\\ &\quad \times\left(L_g(0) + \Delta g(0) - \frac{1}{2}\frac{16}{16+3n_f}\right).\end{aligned} \tag{14}$$

Given a composition of the nucleon spin at some initial scale Q_0^2, the above equations yield the spin composition at any other perturbative scales in the leading-log approximation. From the expression for $L_g(t)$, it is clear that that the large gluon helicity at large Q^2 canceled by an equally large, but negative, gluon orbital angular momentum.

Neglecting the subleading terms at large Q^2, we get

$$\begin{aligned}J_q &= L_q + \tfrac{1}{2}\Delta\Sigma = \tfrac{1}{2}\,3n_f/(16+3n_f)\,,\\ J_g &= L_g + \Delta g = \tfrac{1}{2}\,16/(16+3n_f)\,.\end{aligned} \tag{15}$$

Thus partition of the nucleon spin between quarks and gluons follows the well-known partition of the nucleon momentum [11]. Mathematically, one can understand this from the expression for the QCD angular momentum density $M^{\mu\alpha\beta} = T^{\mu\beta}x^\alpha - T^{\mu\alpha}x^\beta$. When $M^{\mu\alpha\beta}$ and $T^{\alpha\beta}$ are each separated into gluon and quark contributions, the anomalous dimensions of the corresponding terms are the same because they have the same short distance behavior.

It is interesting to speculate phenomenological consequences of this asymptotic partition of the nucleon spin.

Assuming, as found in the case of the momentum sum rule [11], that the evolution in Q^2 is very slow, then the above partition may still be roughly correct at low momentum scales, say, $Q^2 \sim 3$ GeV^2. If this is the case, from the experimentally measured $\Delta\Sigma$ we get an estimate of the quark orbital contribution at these scales,

$$L_q = 0.05\text{–}0.15\,. \tag{16}$$

To find a separation of the gluon contribution into spin and orbit parts, we need to know Δg, which shall be measurable in the future [12]. However, if Q^2 variation is rapid, the asymptotic result implies nothing about the low Q^2 spin structure of the nucleon. Unfortunately, no one knows yet how to measure L_q to determine the role of Q^2 variation.

So far we have considered the leading-log results. It is well known that at the next-to-leading-log level the matrix element of the axial current in a gluon state contains an anomaly [7,13]. Because of this anomaly, the axial current acquires an anomalous dimension starting at the two-loop level [7,14,15]. Also, because of the anomaly, it was suggested that the quark helicity for light flavors be modified to subtract the anomaly contribution [7,13,16]. However, since the axial current appears in the conserved charge J^{12}, the anomaly contribution to $\Delta\Sigma$ must be canceled by matrix elements of some other operators in J^{12}. Below, we show such a cancellation comes from the quark orbital angular momentum $\hat{L}_{3q}$.

To investigate the anomaly cancellation, we consider a gluon moving in the z direction with momentum p^μ and helicity $+1$. The gluon can virtually split into a quark-antiquark pair. For zero quark masses, the quark and antiquark have opposite helicities due to chirality conservation. Thus the helicity of the gluon is entirely transferred to the orbital angular momentum of the pair. However, in the limit of the quark transverse momentum going to infinity, the chiral symmetry is broken by the anomaly. As a consequence, the quark and antiquark can have same helicity. Indeed, the calculation of Carlitz, Collins, and Mueller shows [13]

$$\langle p+| \tfrac{1}{2}\,\overline{\psi}\gamma^+\gamma_5\psi | p+\rangle = -\frac{\alpha_s}{4\pi}\,2p^+\,. \tag{17}$$

The sign indicates that the quark and antiquark prefer to have helicity $-1/2$. To preserve the total angular momentum, the orbital motion of the pair has the same probability to carry two units of angular momentum. Indeed, an explicit calculation yields

$$\langle p+|\hat{L}_{3q}|p+\rangle - \langle p+|p+\rangle = \frac{\alpha_s}{4\pi}\,2p^+(2\pi)^3\delta^3(0)\,. \tag{18}$$

According to the above, if one redefines the quark helicity by subtracting the anomalous gluon contribution [7,13], one must also redefine the quark orbital angular momentum by adding to it the same amount. Thus if a large Δg helps to restore the measured $\Delta\Sigma$ to the quark model result, the same effect will make L_q in Eq. (16) negative. However, the fraction of the nucleon spin carried by quarks is invariant under such redefinition.

To summarize, we have derived an evolution equation for the quark and gluon orbital angular momenta in QCD. In the asymptotic limit, the solution of the equation indicates that the quark and gluon contributions to the nucleon spin are the same as their contributions to the nucleon momentum. Furthermore, both the gluon orbital and helicity contributions grow logarithmically at large Q^2 and with opposite signs, so their sum stays finite. We show that the anomaly in the quark helicity is canceled by a contribution in the quark orbital angular momentum, so the quark contribution to the nucleon spin is anomaly free.

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