Effect of quark antisymmetrization on the binding energy of nuclear matter

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We estimate, to leading order in the nuclear matter density, the effect of antisymmetrizing quarks belonging to different nucleons upon the binding energy per nucleon in nuclear matter. A simple Gaussian model for the nucleon quark wave function, together with a one-gluon-exchange potential between quarks, is assumed. Using a linked cluster expansion developed earlier for nuclear matter, we calculate separately the effect of quark interchanges upon the kinetic, hyperfine, Coulomb, and contact terms in the Hamiltonian. A strong net repulsion is found which grows rapidly with the dimensionless parameter $\alpha = r_0 k_F$, where k_F is the nuclear Fermi momentum and r_0 the nucleon size.

The quark cluster model for nucleons has been used extensively in recent years for estimating nuclear properties associated with short-distance effects in small nuclei. (For a recent review, see Ref. 1.) The necessity of antisymmetrizing the nuclear wave function at the quark level, and the consequences of this for various nuclear observables, has been pointed out by a number of authors.[1-3] Antisymmetrization directly implies that quarks belonging to different nucleons are exchanged in proportion to the degree of nucleon overlap; its importance rises with increasing nuclear density and nucleon size.

In this paper we investigate the effect of quark exchange upon the nuclear binding energy to leading order in the density of spin-isospin-symmetric nuclear matter. A simple Gaussian model for the quarks in a nucleon is assumed, together with a potential arising from one-gluon exchange between quarks. This model, as well as variants of it, have been used by numerous authors in a long series of investigations into the nucleon-nucleon short-range interaction and deuteron properties.[4] The problem of a large nucleus is clearly much more complicated; not only is there a vastly greater number of combinations by which quarks can be exchanged, but also one is faced by certain naive divergences[3] in the limit of large A. To address this issue, a systematic scheme for quark counting, together with elimination of the divergences, was developed in Ref. 3. We shall make essential use of that formalism in this Brief Report.

The model. We take the usual nonrelativistic quark model wave function as an input. A nucleon with center of mass at $\mathbf{R}$ is described by

$$\delta(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3-3\mathbf{R})\exp\left[-\frac{1}{2r_0^2}\sum_{i=1}^{3}(\mathbf{r}_i-\mathbf{R})^2\right]\chi_{\mathrm{STC}}\ , \quad (1)$$

where χ_{STC} is the spin-isospin-color wave function. The Hamiltonian is taken to be

$$H=\sum_i T_i+\tfrac{1}{2}\sum_{ij}(V_{ij}^{\mathrm{conf}}+V_{ij}^{\mathrm{OGE}})\ , \quad (2)$$

where T is the quark kinetic energy, V^{conf} is a harmonic oscillator confining potential which mocks up long-range QCD effects, and V^{OGE} is the quark-quark potential arising from one-gluon exchange,

$$V_{ij}^{\mathrm{OGE}}=V_{ij}^{\mathrm{hyp}}+V_{ij}^{\mathrm{Coul}}+V_{ij}^{\mathrm{cont}}\ . \quad (3a)$$

The hyperfine, Coulomb, and contact terms are, respectively,

$$V_{ij}^{\mathrm{hyp}}=-\frac{8\pi}{3}\frac{\alpha_s}{m^2}\delta(\mathbf{r}_i-\mathbf{r}_j)F_i\cdot F_j\cdot S_i\cdot S_j\ , \quad (3b)$$

$$V_{ij}^{\mathrm{Coul}}=\alpha_s\frac{F_i\cdot F_j}{|\mathbf{r}_i-\mathbf{r}_j|}\ , \quad (3c)$$

$$V_{ij}^{\mathrm{cont}}=-\alpha_s\frac{\pi}{m^2}\delta(\mathbf{r}_i-\mathbf{r}_j)F_i\cdot F_j\ . \quad (3d)$$

The constituent quark mass m is taken to be one-third of the nucleon mass, and α_s is determined by the N-Δ mass difference

$$M_\Delta-M_N=\frac{4}{3}\frac{\alpha_s}{\sqrt{2\pi}m^2r_0^3}\ . \quad (4)$$

The nucleon rms radius r_0 will be taken to be a free parameter in our calculation. We have made no attempt to include the long-range part of the nucleon-nucleon force arising from meson exchange into our calculations.

To enforce antisymmetrization at the quark level it is convenient to work in a second quantized formalism as in Ref. 3 and to write the N nucleon state as

$$|A\rangle=\frac{1}{\sqrt{A!}}\Phi^{\alpha_1,\ldots,\alpha_N}A_{\alpha_1}^{\dagger}\cdots A_{\alpha_N}^{\dagger}|0\rangle\ , \quad (5)$$

where Φ is the nuclear wave function and α_i denotes the nucleon quantum numbers (spin, isospin, and position of c.m.). The nucleon creation operator $A^{\dagger}\sim q^{\dagger}q^{\dagger}q^{\dagger}$ creates a three-quark state coupled together by the wave function given in Eq. (1) to the appropriate nucleon quantum numbers. The Hamiltonian Eq. (2). is a sum of operators of the type $q^{\dagger}q$ (kinetic energy) and $q^{\dagger}q^{\dagger}qq$ (potential energy). We now consider evaluation of these separately.

Kinetic energy. The expectation of the quark kinetic energy in the nuclear state $|A\rangle$ is

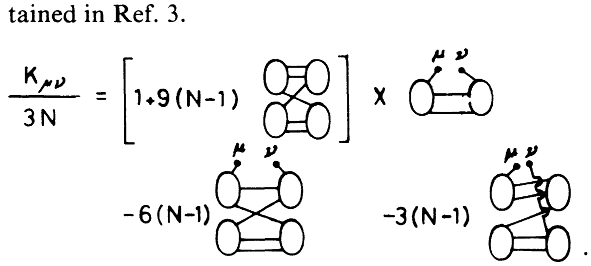
$$\langle T\rangle=\sum_{\mu\nu}\langle\mu|T|\nu\rangle K_{\mu\nu}\ , \tag{6a}$$

where

$$K_{\mu\nu}=\frac{\langle A|q_\mu^\dagger q_\nu|A\rangle}{\langle A|A\rangle}\ . \tag{6b}$$

Here μ or ν denote a set of quark quantum numbers: position, spin, isospin, and color.

Evaluation of $K_{\mu\nu}$ requires the complete contraction of $3N+1$ quark creation operators with $3N+1$ destruction operators. This corresponds to quarks being exchanged between the N nucleons in $(3N+1)!$ ways. To make the calculation tractable, simultaneous quark exchanges between three or more nucleons will be neglected. Nevertheless, there is no constraint on the number of nucleon pairs that can exchange quarks simultaneously (within each pair). This causes both the numerator and denominator in $K_{\mu\nu}$ to diverge separately with N, and consequently requires renormalization of $K_{\mu\nu}$. The following renormalized result, expressed diagrammatically, was obtained in Ref. 3.



(7)

The diagrammatic representation above is directly translatable into mathematical expressions,[3] and represents the renormalized sum of zero and one quark exchanges that enter into the evaluation of an arbitrary one-quark operator.

To evaluate $\langle T\rangle$ we need a model for the nuclear wave function Φ. In the first instance, this was taken to be a Slater determinant of plane waves containing equal numbers of spin-up and spin-down neutrons and protons. A lengthy evaluation, using the Gaussian nucleon wave function in Eq. (1) and the renormalized expression in Eq. (7), and after performing spin-isospin-color averages, yields the following remarkably simple result for the kinetic energy per nucleon in nuclear matter:

$$\begin{aligned}\frac{\langle T\rangle}{N}&=\frac{3}{2mr_0^2}+\frac{3}{5}\frac{k_F^2}{2m}-\tfrac{1}{16}\sqrt{3/\pi}\,\frac{k_F^3r_0}{m}+O(k_F^5)\\ &=\frac{1}{2mr_0^2}\left[\frac{3}{2}+\frac{3\alpha^2}{10}-\frac{1}{16}\sqrt{3/\pi}\,\alpha^3+O(\alpha^5)\right]\end{aligned} \tag{8}$$

where $\alpha=k_Fr_0$ is a dimensionless quantity. The first two terms in Eq. (8) are familiar; they represent, respectively, the mean kinetic energy of quarks within a stationary nucleon and the averaged kinetic energy of the c.m. motion of nucleons in a free Fermi gas. The remaining term, i.e.,

$$\langle T\rangle_{\text{exch}}=-\frac{1}{16}\sqrt{3/\pi}\,\frac{\alpha^3}{mr_0^2} \tag{9}$$

is the quark exchange correction to first order in the nuclear matter density. The reduction in kinetic energy is equivalent to a softening of the quark momentum distribution. That quark interchange does lead to such softening had been noted earlier by Hoodbhoy and Jaffe.[2]

An uncorrelated Fermi gas is rather unrealistic for nuclear matter and the repulsion between nucleons prevents them from overlapping. We include correlations, admittedly in a rather crude way, by simply cutting off all integrals in the evaluation of Eq. (7) whenever the distance between the centers of mass of two nucleons is less than the hard-core radius r_c. This, expectedly, decreases the overlap between nucleons and reduces the exchange kinetic energy:

$$\langle T\rangle_{\text{exch}}=-\frac{1}{16}\sqrt{3/\pi}\,\frac{\alpha^3}{mr_0^2}f\left[\frac{r_c}{r_0}\right]\ , \tag{10a}$$

where

$$f(s)=\frac{2\pi}{3}\int_s^\infty\int_s^\infty dx\,dy\,xy\times\exp[-\tfrac{15}{16}(x^2+y^2)]\sinh\tfrac{9}{8}xy\ . \tag{10b}$$

It can be seen from the above expression that $f(0)=1$, and that $f(s)$ decreases rapidly with s. The exchange kinetic energy as a function of nucleon size, and for different hard-core radii R_c, is given in Fig. 1.

Potential energy. The potential energy of quarks in the nuclear state $|A\rangle$ is

$$\langle V\rangle=\tfrac{1}{2}\sum\langle\mu_1\mu_2|V|\nu_1\nu_2\rangle K_{\mu_1\mu_2,\nu_1\nu_2}\ , \tag{11a}$$

where

$$K_{\mu_1\mu_2,\nu_1\nu_2}=\frac{\langle A|q_{\mu_1}^\dagger q_{\mu_2}^\dagger q_{\nu_2}q_{\nu_1}|A\rangle}{\langle A|A\rangle}\ . \tag{11b}$$

Dropping gluon exchange diagrams that vanish because of color conservation, the mean potential energy per quark is calculated from the following equation.

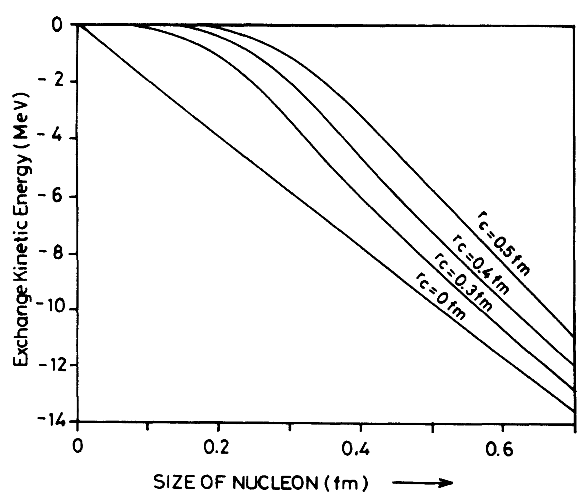


FIG. 1. Quark exchange contribution to quark kinetic energy for spin-isospin-symmetric nuclear matter as a function of nucleon size for different hard-core radii and $k_F=1.4\ \text{fm}^{-1}$.

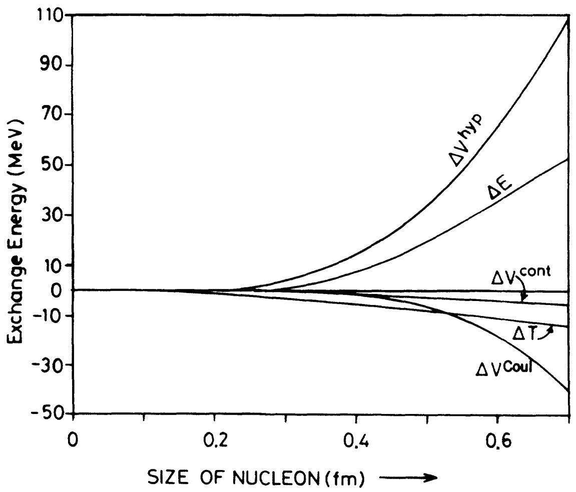
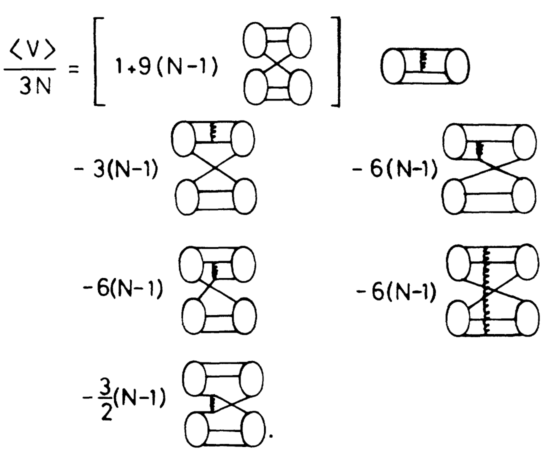


FIG. 2. Quark exchange contribution to the kinetic, hyperfine, Coulomb, and contact terms for hard-core radius $r_C = 0.3$ fm and $k_F = 1.4$ fm^{-1}.



(12)

The weights in Eq. (12) follow directly from the general renormalized expansion for $K_{\mu_1\mu_2,\nu_1\nu_2}$, which was developed in Ref. 3. The evaluation of the terms in this equation, using the potential in Eq. (3), follows the same procedure as for the kinetic energy but is considerably more involved. The result of the calculation has the general form

$$\langle V \rangle_{\text{exch}} = \langle V^{\text{hyp}} \rangle_{\text{exch}} + \langle V^{\text{Coul}} \rangle_{\text{exch}} + \langle V^{\text{cont}} \rangle_{\text{exch}}$$
$$= \frac{\alpha_s a^3}{m^2 r_0^3} g_1 \left[\frac{r_c}{r_0} \right] - \frac{\alpha_s a^3}{r_0} g_2 \left[\frac{r_c}{r_0} \right] - \frac{\alpha_s a^3}{m^2 r_0^3} g_3 \left[\frac{r_c}{r_0} \right] . \quad (13)$$

The expressions for the functions g_i are complicated and rather unrevealing. Hence we give, in Figs. 2 and 3, only numerical plots for the hyperfine, Coulomb, and contact exchange energies for two different hard-core radii. We note that the confinement term has no exchange piece.

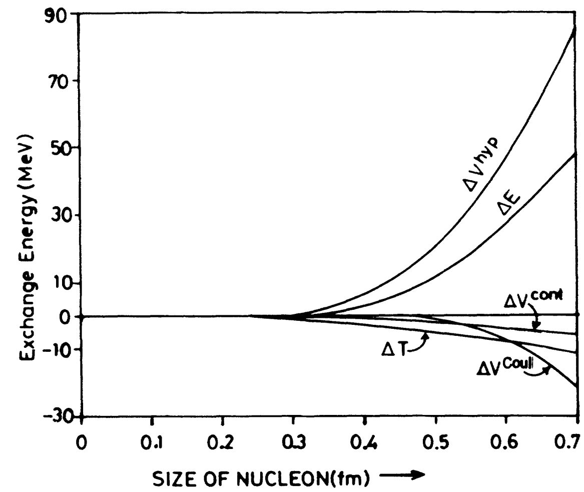


FIG. 3. Quark exchange contribution to the kinetic, hyperfine, Coulomb, and contact terms for hard-core radius $r_c = 0.5$ fm and $k_F = 1.4$ fm^{-1}.

Discussion. The net result of our calculations, i.e., the sum of the kinetic and potential energy contributions that arise from the exchange of quarks between nucleons inside spin-isospin-symmetric nuclear matter, is summarized in Figs. 2, 3. As can be seen, the net effect of quark exchange is to introduce a strong repulsion. The dominant contribution comes from the hyperfine term in the Hamiltonian.

The calculations outlined in this paper can be repeated for any value of N. The case of the deuteron, $N=2$, is particularly interesting and easy. As the input wave function we used a pure s-wave sum of Gaussians,[5]

$$\Phi(R) = N[e^{-0.02R^2} + 2.8e^{-0.10125R^2} - 3.4e^{-1.28R^2}] . \quad (14)$$

The quark exchange contributions to the kinetic and potential terms are plotted in Fig. 4. Since the deuteron is a

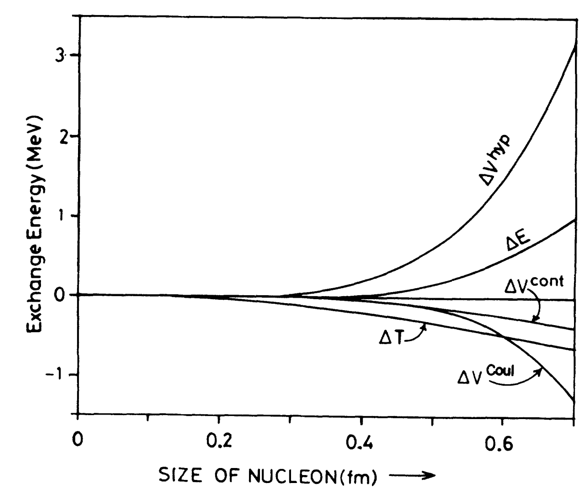


FIG. 4. Quark exchange contribution to the kinetic, hyperfine, Coulomb, and contact terms for the deuteron, using the wave function in Eq. (14).

very dilute system, these are much smaller than in nuclear matter. Nevertheless, we still observe that the exchange hyperfine interaction dominates.

The repulsion induced by the hyperfine interaction can be seen in rather dramatic terms if one calculates the energy of a six-quark (6q) cluster with all quarks in the lowest s state relative to the energy of two separated three-quark (3q) clusters. The result is

$$\langle V^{\rm hyp}(6q)\rangle - 2\langle V^{\rm hyp}(3q)\rangle = 1.29(M_\Delta - M_N)\ . \quad (15)$$

This shows that it is energetically highly preferable for a set of six quarks to disassociate into two nucleons. The importance of the hyperfine interaction in this regard has been recognized by many authors.[6]

ACKNOWLEDGMENT

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