Chapter Eight

Non-linear oscillations and phase space

KEY FEATURES

The key features of this chapter are the use of **perturbation theory** to solve weakly non-linear problems, the notion of **phase space**, the **Poincaré–Bendixson** theorem, and **limit cycles**.

In reality, most oscillating mechanical systems are governed by **non-linear equations**. The linear oscillation theory developed in Chapter 5 is generally an approximation which is accurate only when the amplitude of the oscillations is small. Unfortunately, non-linear oscillation equations do not have nice exact solutions as their linear counterparts do, and this makes the non-linear theory difficult to investigate analytically.

In this chapter we describe two different analytical approaches, each of which is successful in its own way. The first is to use **perturbation theory** to find successive corrections to the linear theory. This gives a more accurate solution than the linear theory when the non-linear terms in the equation are small. However, because the solution is close to that predicted by the linear theory, new phenomena associated with non-linearity are unlikely to be discovered by perturbation theory! The second approach involves the use of geometrical arguments in **phase space**. This has the advantage that the non-linear effects can be large, but the conclusions are likely to be qualitative rather than quantitative. A particular triumph of this approach is the **Poincaré–Bendixson** theorem, which can be used to prove the existence of **limit cycles**, a new phenomenon that exists only in the non-linear theory.

8.1 PERIODIC NON-LINEAR OSCILLATIONS

Most oscillating mechanical systems are not exactly linear but are approximately linear when the oscillation amplitude is small. In the case of a body on a spring, the restoring force might actually have the form

$$S = m\Omega^2 x + m\Lambda x^3, \tag{8.1}$$

which is approximated by the linear formula $S = m\Omega^2 x$ when the displacement x is small. The new constant Λ is a measure of the strength of the non-linear effect. If $\Lambda < 0$, then

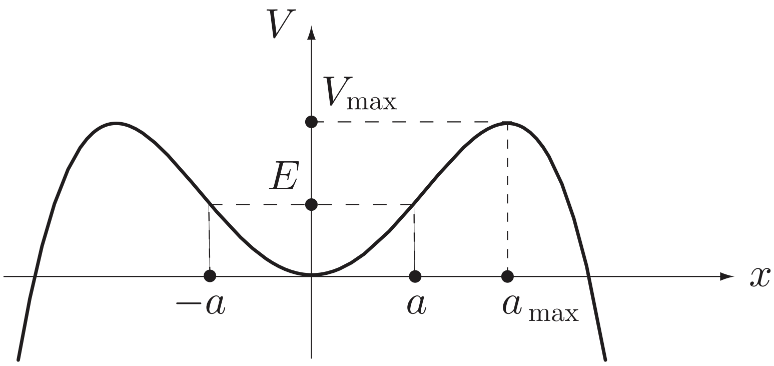

FIGURE 8.1 Existence of periodic oscillations for the quartic potential energy $V = \frac{1}{2}m\Omega^2x^2 + \frac{1}{4}m\Lambda x^4$ with $\Lambda < 0$.

S is less than its linear approximation and the spring is said to be **softening** as x increases. Conversely, if $\Lambda > 0$, then the spring is **hardening** as x increases. The formula (8.1) is typical of non-linear restoring forces that are *symmetrical* about $x = 0$. If the restoring force is unsymmetrical about $x = 0$, the leading correction to the linear case will be a term in x^2.

Existence of non-linear periodic oscillations

Consider the **free undamped oscillations** of a body sliding on a smooth horizontal table* and connected to a fixed point of the table by a spring whose restoring force is given by the cubic formula (8.1). In rectilinear motion, the **governing equation** is then

$$\frac{d^2x}{dt^2} + \Omega^2x + \Lambda x^3 = 0, \tag{8.2}$$

which is Duffing's equation with no forcing term (see section 8.5). The existence of periodic oscillations can be proved by the energy method described in Chapter 6. The restoring force has potential energy

$$V = \tfrac{1}{2}m\Omega^2x^2 + \tfrac{1}{4}m\Lambda x^4,$$

so that the energy conservation equation is

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}m\Omega^2x^2 + \tfrac{1}{4}m\Lambda x^4 = E,$$

where $v = \dot{x}$. The motion is therefore restricted to those values of x that satisfy

$$\tfrac{1}{2}m\Omega^2x^2 + \tfrac{1}{4}m\Lambda x^4 \le E,$$

* Would the motion be the same (relative to the equilibrium position) if the body were suspended vertically by the same spring?

with equality when $v = 0$. Figure 8.1 shows a sketch of V for a *softening* spring ($\Lambda < 0$). For each value of E in the range $0 < E < V_{\max}$, the particle oscillates in a symmetrical range $-a \le x \le a$ as shown. Thus oscillations of any amplitude less than $a_{\max}$ ($= \Omega/|\Lambda|^{1/2}$) are possible. For a *hardening* spring, oscillations of any amplitude whatsoever are possible.

Solution by perturbation theory

Suppose then that the body is performing periodic oscillations with amplitude a. In order to reduce the number of parameters, we non-dimensionalise equation (8.2). Let the *dimensionless displacement* X be defined by $x = aX$. Then X satisfies the equation

$$\frac{1}{\Omega^2}\frac{d^2X}{dt^2} + X + \epsilon X^3 = 0, \tag{8.3}$$

with the initial conditions $X = 1$ and $dX/dt = 0$ when $t = 0$. The dimensionless parameter ϵ, defined by

$$\epsilon = \frac{a^2\Lambda}{\Omega^2}, \tag{8.4}$$

is a measure of the strength of the non-linearity. Equation (8.3) contains ϵ as a parameter and hence so does the solution. A major feature of interest is how the period τ of the motion varies with ϵ.

The non-linear equation of motion (8.3) cannot be solved explicitly but it reduces to a simple linear equation when the parameter ϵ is zero. In these circumstances, one can often find an *approximate* solution to the non-linear equation *valid when ϵ is small*. Equations in which the non-linear terms are small are said to be **weakly non-linear** and the solution technique is called **perturbation theory**. There is a well established theory of such perturbations. The simplest case is as follows:

Regular perturbation expansion

If the parameter ϵ appears as the coefficient of any term of an ODE that is *not* the highest derivative in that equation, then, when ϵ is small, the solution corresponding to fixed initial conditions can be expanded as a **power series** in ϵ.

This is called a **regular perturbation expansion*** and it applies to the equation (8.3). It follows that the solution $X(t, \epsilon)$ can be expanded in the **regular perturbation series**

$$X(t, \epsilon) = X_0(t) + \epsilon X_1(t) + \epsilon^2 X_2(t) + \cdots . \tag{8.5}$$

* The case in which the small parameter multiplies the *highest* derivative in the equation is called a **singular perturbation**. For experts only!

The standard method is to substitute this series into the equation (8.3) and then to try to determine the functions $X_0(t)$, $X_1(t)$, $X_2(t)$, $\dots$. In the present case however, this leads to an unsatisfactory result because the functions $X_1(t)$, $X_2(t)$, $\dots$, turn out to be *non-periodic* (and unbounded) even though the exact solution $X(t,\epsilon)$ is periodic!* Also, it is not clear how to find approximations to τ from such a series.

This difficulty can be overcome by replacing t by a new variable s so that the solution $X(s,\epsilon)$ has period 2π in s *whatever the value of* ϵ. Every term of the perturbation series will then also be periodic with period 2π. This trick is known as *Lindstedt's method*.

Lindstedt's method

Let $\omega(\epsilon)$ $(= 2\pi/\tau(\epsilon))$ be the angular frequency of the required solution of equation (8.3). Now introduce a new independent variable s (the *dimensionless time*) by the equation $s = \omega(\epsilon)t$. Then $X(s,\epsilon)$ satisfies the equation

$$\left(\frac{\omega(\epsilon)}{\Omega}\right)^2 X'' + X + \epsilon X^3 = 0 \tag{8.6}$$

with the initial conditions $X = 1$ and $X' = 0$ when $s = 0$. (Here $'$ means d/ds.) We now seek a solution of this equation in the form of the perturbation series

$$X(s,\epsilon) = X_0(s) + \epsilon X_1(s) + \epsilon^2 X_2(s) + \cdots . \tag{8.7}$$

which is possible when ϵ is small. By construction, this solution must have period 2π *for all* ϵ from which it follows that each of the functions $X_0(s)$, $X_1(s)$, $X_2(s)$, $\dots$ must also have period 2π. However we have paid a price for this simplification since the *unknown* angular frequency $\omega(\epsilon)$ now appears in the equation (8.6); indeed, the function $\omega(\epsilon)$ is part of the *answer* to this problem! We must therefore also expand $\omega(\epsilon)$ as a perturbation series in ϵ. From equation (8.3), it follows that $\omega(0) = \Omega$ so we may write

$$\frac{\omega(\epsilon)}{\Omega} = 1 + \omega_1\epsilon + \omega_2\epsilon^2 + \cdots , \tag{8.8}$$

where $\omega_1, \omega_2, \dots$ are unknown constants that must be determined along with the functions $X_0(s)$, $X_1(s)$, $X_2(s)$, $\dots$.

On substituting the expansions (8.7) and (8.8) into the governing equation (8.6) and its initial conditions, we obtain:

$$(1 + \omega_1\epsilon + \omega_2\epsilon^2 + \cdots)^2(X_0'' + \epsilon X_1'' + \epsilon^2 X_2'' + \cdots) +$$
$$(X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots) + \epsilon(X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots)^3 = 0,$$

* This 'paradox' causes great bafflement when first encountered, but it is inevitable when the period τ of the motion depends on ϵ, as it does in this case. To have a series of non-periodic terms is not *wrong*, as is sometimes stated. However, it is certainly unsatisfactory to have a non-periodic approximation to a periodic function.

with

$$X_0 + \epsilon X_1 + \epsilon^2 X_2 + \cdots = 1,$$
$$X_0' + \epsilon X_1' + \epsilon^2 X_2' + \cdots = 0,$$

when $s = 0$. If we now equate coefficients of powers of ϵ in these equalities, we obtain a succession of ODEs and initial conditions, the first two of which are as follows:

From coefficients of ϵ^0, we obtain the **zero order** equation

$$X_0'' + X_0 = 0, \tag{8.9}$$

with $X_0 = 1$ and $X_0' = 0$ when $s = 0$.
From coefficients of ϵ^1, we obtain the **first order** equation

$$X_1'' + X_1 = -2\omega_1 X_0'' - X_0^3, \tag{8.10}$$

with $X_1 = 0$ and $X_1' = 0$ when $s = 0$.

This procedure can be extended to any number of terms but the equations rapidly become very complicated. The method now is to solve these equations in order; the only sticking point is how to determine the unknown constants $\omega_1, \omega_2, \ldots$ that appear on the right sides of the equations. The solution of the **zero order** equation and initial conditions is

$$X_0 = \cos s \tag{8.11}$$

and this can now be substituted into the first order equation (8.10) to give

$$\begin{aligned} X_1'' + X_1 &= 2\omega_1 \cos s - \cos^3 s \\ &= \tfrac{1}{4}\,(8\omega_1 - 3)\cos s + \tfrac{1}{4}\cos 3s, \end{aligned} \tag{8.12}$$

on using the trigonometric identity $\cos 3s = 4\cos^3 s - 3\cos s$. This equation can now be solved by standard methods. The particular integral corresponding to the $\cos 3s$ on the right is $-(1/8)\cos 3s$, but the particular integral corrsponding to the $\cos s$ on the right is $(1/2)s \sin s$, since $\cos s$ is a solution of the equation $X'' + X = 0$. The general solution of the first order equation is therefore

$$X_1 = \left(\omega_1 - \tfrac{3}{8}\right) s \sin s - \tfrac{1}{32}\cos 3s + A\cos s + B\sin s,$$

where A and B are arbitrary constants. Observe that the functions $\cos s$, $\sin s$ and $\cos 3s$ are all periodic with period 2π, but the term $s \sin s$ is *not periodic*. Thus, *the coefficient of $s \sin s$ must be zero, for otherwise $X_1(s)$ would not be periodic, which we know it must be*. Hence

$$\omega_1 = \frac{3}{8}, \tag{8.13}$$

which determines the first unknown coefficient in the expansion (8.8) of $\omega(\epsilon)$. The solution of the **first order** equation and initial conditions is then

$$X_1 = \frac{1}{32}\left(\cos s - \cos 3s\right). \tag{8.14}$$

We have thus shown that, when ϵ is small,

$$\frac{\omega}{\Omega} = 1 + \frac{3}{8}\epsilon + O\left(\epsilon^2\right),$$

and

$$X = \cos s + \frac{\epsilon}{32}\left(\cos s - \cos 3s\right) + O\left(\epsilon^2\right),$$

where $s = \left(1 + \frac{3}{8}\epsilon + O\left(\epsilon^2\right)\right)\Omega t$.

Results

When ϵ $(= a^2\Lambda/\Omega^2)$ is small, the **period** τ of the oscillation of equation (8.2) with amplitude a is given by

$$\tau = \frac{2\pi}{\omega} = \frac{2\pi}{\Omega}\left(1 + \frac{3}{8}\epsilon + O\left(\epsilon^2\right)\right)^{-1} = \frac{2\pi}{\Omega}\left(1 - \frac{3}{8}\epsilon + O\left(\epsilon^2\right)\right) \tag{8.15}$$

and the corresponding displacement $x(t)$ is given by

$$x = a\left[\cos s + \frac{\epsilon}{32}\left(\cos s - \cos 3s\right) + O\left(\epsilon^2\right)\right], \tag{8.16}$$

where $s = \left(1 + \frac{3}{8}\epsilon + O\left(\epsilon^2\right)\right)\Omega t$.

This is the *approximate solution correct to the first order in the small parameter* ϵ. More terms can be obtained in a similar way but the effort needed increases exponentially and this is best done with computer assistance (see Problem 8.15).

These formulae apply only when ϵ is small, that is, when the *non- linearity in the equation has a small effect*. Thus we have laboured through a sizeable chunk of mathematics to produce an answer that is only slightly different from the linear case. This sad fact is true of all *regular* perturbation problems. However, in non-linear mechanics, one must be thankful for even modest successes.

8.2 THE PHASE PLANE ((x_1, x_2)–plane)

The second approach that we will describe could not be more different from perturbation theory. It makes use of qualitative geometrical arguments in the phase space of the system.

Systems of first order ODEs

The notion of **phase space** springs from the theory of **systems of first order ODEs**. Such systems are very common and need have no connection with classical mechanics. A standard example is the predator-prey system of equations

$$\begin{aligned}\dot{x}_1 &= ax_1 - bx_1x_2,\\ \dot{x}_2 &= bx_1x_2 - cx_2,\end{aligned}$$

which govern the population density $x_1(t)$ of a prey and the population density $x_2(t)$ of its predator. In the general case there are n unknown functions satisfying n first order ODEs, but here we will only make use of *two* unknown functions $x_1(t)$, $x_2(t)$ that satisfy a *pair* of first order ODEs of the form

$$\begin{aligned}\dot{x}_1 &= F_1(x_1, x_2, t),\\ \dot{x}_2 &= F_2(x_1, x_2, t).\end{aligned} \tag{8.17}$$

Just to confuse matters, a system of ODEs like (8.17) is called a **dynamical system**, whether it has any connection with classical mechanics or not! In the predator-prey dynamical system, the function $F_1 = ax_1 - bx_1x_2$ and the function $F_2 = bx_1x_2 - cx_2$. In this case F_1 and F_2 have no explicit time dependence. Such systems are said to be *autonomous*; as we shall see, more can be said about the behaviour of autonomous systems.

Definition 8.1 ***Autonomous system*** *A system of equations of the form*

$$\begin{aligned}\dot{x}_1 &= F_1(x_1, x_2),\\ \dot{x}_2 &= F_2(x_1, x_2),\end{aligned} \tag{8.18}$$

is said to be ***autonomous****.*

The phase plane

The values of the variables x_1, x_2 at any instant can be represented by a point in the (x_1, x_2)-plane. This plane is called the **phase plane*** of the system. A solution of the system of equations (8.17) is then represented by a point moving in the phase plane. The path traced out by such a point is called a **phase path**† of the system and the set of all phase paths is called the **phase diagram**. In the predator–prey problem, the variables x_1, x_2 are positive quantities and so the physically relevant phase paths lie in the first quadrant of the phase plane. It can be shown that they are all closed curves! (See Problem 8.10).

Phase paths of autonomous systems

The problem of finding the phase paths is much easier when the system is **autonomous**. The method is as follows:

* In the general case with n unknowns, the phase space is n-dimensional.

† Also called an *orbit* of the system.

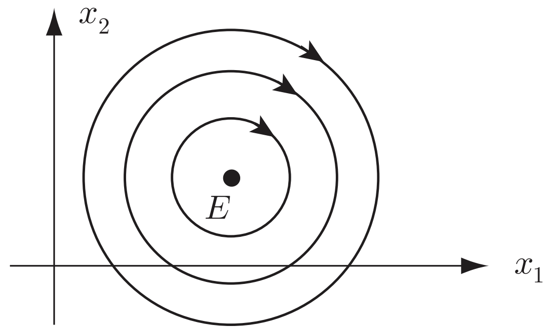

FIGURE 8.2 Phase diagram for the system $dx_1/dt = x_2 - 1,\ dx_2/dt = -x_1 + 2$. The point $E(2, 1)$ is an equilibrium point of the system.

Example 8.1 *Finding phase paths for an autonomous system*

Sketch the phase diagram for the autonomous system of equations

$$\frac{dx_1}{dt} = x_2 - 1,$$

$$\frac{dx_2}{dt} = -x_1 + 2.$$

Solution

The phase paths of an *autonomous* system can be found by eliminating the time derivatives. The path gradient is given by

$$\frac{dx_2}{dx_1} = \frac{dx_2/dt}{dx_1/dt} = -\frac{x_1 - 2}{x_2 - 1}$$

and this is a first order separable ODE satisfied by the phase paths. The general solution of this equation is

$$(x_1 - 2)^2 + (x_2 - 1)^2 = C$$

and each (positive) choice for the constant of integration C corresponds to a phase path. The **phase paths** are therefore circles with centre (2, 1); the **phase diagram** is shown in Figure 8.2.

The direction in which the phase point progresses along a path can be deduced by examining the *signs* of the right sides in equations (8.18). This gives the signs of $\dot{x}_1$ and $\dot{x}_2$ and hence the direction of motion of the phase point. ■

When the system is autonomous, one can say quite a lot about the *general nature* of the phase paths without finding them. The basic result is as follows:

Theorem 8.1 ***Autonomous systems: a basic result*** *Each point of the phase space of an autonomous system has exactly one phase path passing through it.*

Proof. Let (a, b) be any point of the phase space. Suppose that the motion of the phase point (x_1, x_2) satisfies the equations (8.18) and that the phase point is at (a, b) when $t = 0$. The general theory of ODEs then tells us that a solution of the equations (8.18), that satisfies the initial conditions $x_1 = a$, $x_2 = b$ when $t = 0$, exists and is unique. Let this solution be $\{X_1(t), X_2(t)\}$, which we will suppose is defined for all t, both positive and negative. This phase path certainly passes through the point (a, b) and we must now show that there is no other. Suppose then that there is another solution of the equations in which the phase point is at (a, b) when $t = \tau$, say. This motion also exists and is uniquely determined and, in the general case, would not be related to $\{X_1(t), X_2(t)\}$. However, for autonomous systems, the right sides of equations (8.18) are independent of t so that *the two motions differ only by a shift in the origin of time*. To be precise, the new motion is simply $\{X_1(t - \tau), X_2(t - \tau)\}$. Thus, although the two motions are distinct, the two phase points travel along the *same path* with the second point delayed relative to the first by the constant time τ. Hence, although there are infinitely many *motions* of the phase point that pass through the point (a, b), they all follow the same path. This proves the theorem. ■

Some important deductions follow from this basic result.

Phase paths of autonomous systems

- Distinct phase paths of an autonomous system **do not cross** or touch each other.
- **Periodic motions** of an autonomous system correspond to phase paths that are simple* **closed loops**.

Figure 8.2 showsthe phase paths of an autonomous system. For this system, *all* of the phase paths are simple closed loops and so every motion is periodic. An exception occurs if the phase point is started from the point (2, 1). In this case the system has the constant solution $x_1 = 2$, $x_2 = 1$ so that the phase point never moves; for this reason, the point (2, 1) is called an **equilibrium point** of the system. In this case, the 'path' of the phase point consists of the *single point* (2, 1). However, this still qualifies as a path and the above theory still applies. Consequently no 'real' path may pass through an equilibrium point of an autonomous system.†

8.3 THE PHASE PLANE IN DYNAMICS ((x, v)–plane)

The above theory seems unconnected to classical mechanics since dynamical equations of motion are *second order* ODEs. However, *any second order ODE can be expressed as a pair of first order ODEs*. For example, consider the general linear oscillator equation

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \Omega^2 x = F(t). \tag{8.19}$$

If we introduce the new variable $v = dx/dt$, then

$$\frac{dv}{dt} + 2kv + \Omega^2 x = F(t).$$

* A simple curve is one that does not cross (or touch) itself (except possibly to close).

† It may appear from diagrams that phase paths *can* pass through equilibrium points. This is not so. Such a path approaches *arbitrarily close* to the equilibrium point in question, but never reaches it!

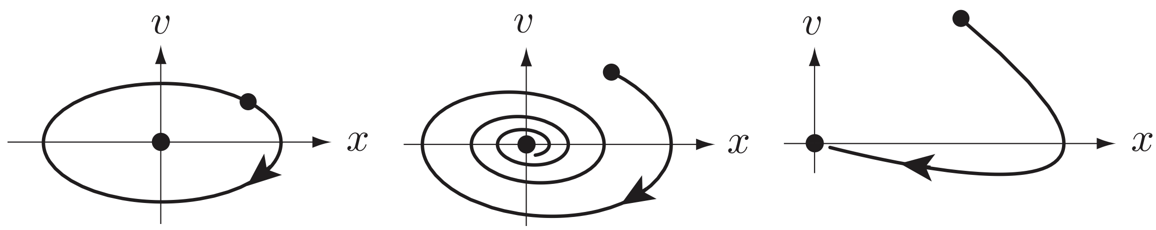

FIGURE 8.3 Typical phase paths for the simple harmonic oscillator equation. **Left**: No damping. **Centre**: Sub-critical damping. **Right**: Super-critical damping.

It follows that the second order equation (8.19) is equivalent to the pair of first order equations

$$\frac{dx}{dt} = v,$$
$$\frac{dv}{dt} = F(t) - 2kv - \Omega^2 x.$$

We may now apply the theory we have developed to this system of first order ODEs, where the **phase plane** is now the (x, v)-plane. It is clear that **driven motion** leads to a **non-autonomous system** because of the presence of the explicit time dependence of $F(t)$; **undriven motion** (in which $F(t) = 0$) leads to an **autonomous** system. It is also clear that equilibrium points in the (x, v)-plane lie on the x-axis and correspond to the ordinary equilibrium positions of the particle.

The form of the phase paths for the *undriven* SHO equation

$$\frac{d^2x}{dt^2} + 2K\frac{dx}{dt} + \Omega^2 x = 0$$

depends on the parameters K and Ω. We could find these paths by the method used in Example 8.1, but there is no point in doing so since we have already solved the equation explicitly in Chapter 5. For instance, when $K = 0$, the general solution is given by

$$x = C\cos(\Omega t - \gamma),$$

from which it follows that

$$v = \frac{dx}{dt} = -C\Omega\sin(\Omega t - \gamma).$$

The phase paths in the (x, v)-plane are therefore similar ellipses centred on the origin, which is an equilibrium point. This, and two typical cases of damped motion, are shown in Figure 8.3. In the presence of damping, the phase point *tends* to the equilibrium point at the origin as $t \to \infty$. Although the equilibrium point is never actually reached, it is convenient to say that these paths 'terminate' at the origin.

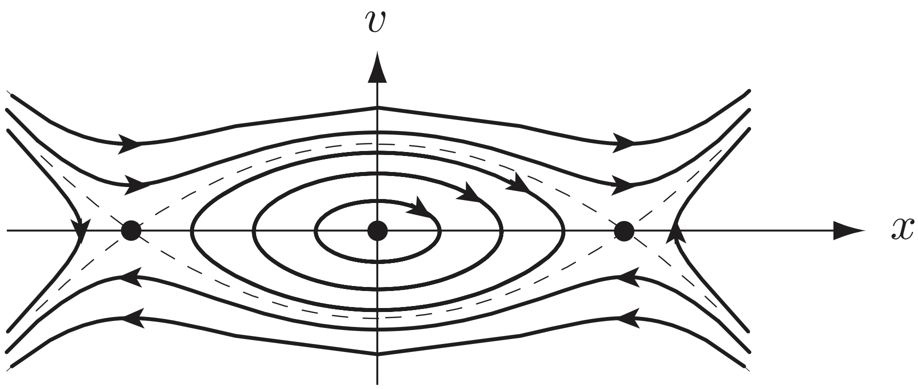

FIGURE 8.4 The phase diagram for the undamped Duffing equation with a softening spring.

Example 8.2 *Phase diagram for equation* $d^2x/dt^2 + \Omega^2 x + \Lambda x^3 = 0$

Sketch the phase diagram for the non-linear oscillation equation

$$d^2x/dt^2 + \Omega^2 x + \Lambda x^3 = 0,$$

when $\Lambda < 0$ (the softening spring).

Solution

This equation is equivalent to the pair of first order equations

$$\frac{dx}{dt} = v,$$
$$\frac{dv}{dt} = -\Omega^2 x - \Lambda x^3,$$

which is an **autonomous** system. The **phase paths** satisfy the equation

$$\frac{dv}{dx} = -\frac{\Omega^2 x + \Lambda x^3}{v},$$

which is a first order separable ODE whose general solution is

$$v^2 = C - \Omega^2 x^2 - \tfrac{1}{2}\Lambda x^4,$$

where C is a constant of integration. Each *positive* value of C corresponds to a phase path. The phase diagram for the case $\Lambda < 0$ is shown in Figure 8.4. There are three **equilibrium points** at $(0, 0)$, $(\pm\Omega/|\Lambda|^{1/2}, 0)$. The closed loops around the origin correspond to **periodic oscillations** of the particle about $x = 0$. Such oscillations can therefore exist for any amplitude less than $\Omega/|\Lambda|^{1/2}$; this confirms the prediction of the energy argument used earlier. Outside this region of closed loops, the paths are unbounded and correspond to unbounded motions of the particle. These two regions of differing behaviour are separated by the dashed paths (known as *separatrices*) that 'terminate' at the equilibrium points $(\pm\Omega/|\Lambda|^{1/2}, 0)$. ■

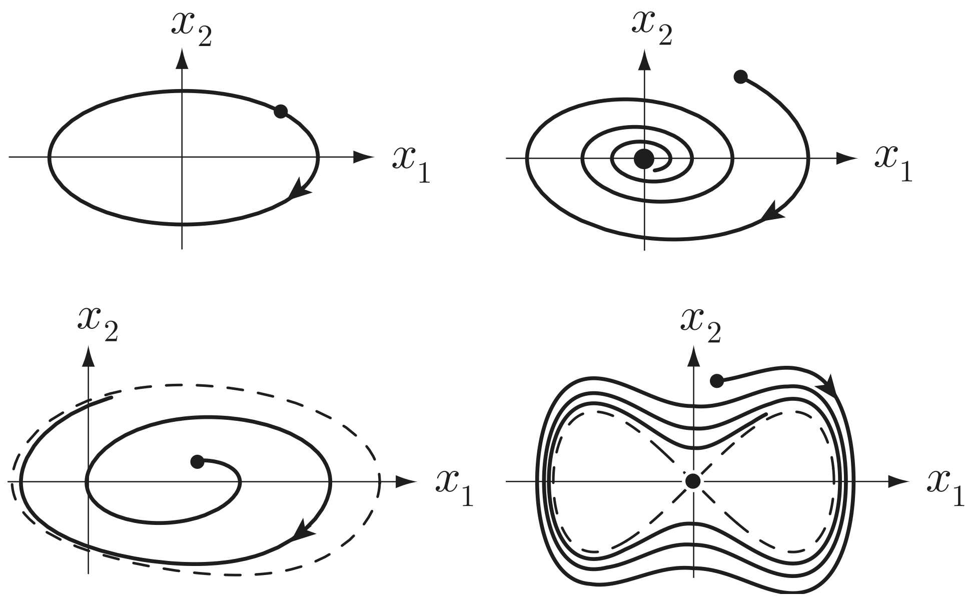

FIGURE 8.5 The Poincaré–Bendixson theorem. Any *bounded* phase path of a plane autonomous system must either close itself (**top left**), terminate at an equilibrium point (**top right**), or tend to a limit cycle (normal case **bottom left**, degenerate case **bottom right**).

8.4 POINCARÉ–BENDIXSON THEOREM: LIMIT CYCLES

In the autonomous systems we have studied so far, those phase paths that are *bounded* either (i) form a closed loop (corresponding to periodic motion), or (ii) 'terminate' at an equilibrium point (so that the motion dies away). Figure 8.3 shows examples of this. The famous Poincaré–Bendixson theorem* which is stated below, says that there is just *one* further possibility.

Poincaré–Bendixson theorem

Suppose that a phase path of a **plane autonomous system** lies in a **bounded** domain of the phase plane for $t > 0$. Then the path must either

- **close** itself, or
- **terminate** at an equilibrium point as $t \to \infty$, or
- tend to a **limit cycle** (or a degenerate limit cycle) as $t \to \infty$.

A proper proof of the theorem is long and difficult (see Coddington & Levinson [9]).

* After Jules Henri Poincaré (1854–1912) and Ivar Otto Bendixson (1861–1935). The theorem was first proved by Poincaré but a more rigorous proof was given later by Bendixson.

The third possibility is new and needs explanation. A **limit cycle** is a **periodic motion** of a special kind. It is *isolated* in the sense that nearby phase paths are *not* closed but are attracted towards the limit cycle* ; they spiral around it (or inside it) getting ever closer, as shown in Figure 8.5 (bottom left). The *degenerate* limit cycle shown in Figure 8.5 (bottom right) is an obscure case in which the limiting curve is not a periodic motion but has one or more equilibrium points actually on it. This case is often omitted in the literature, but it definitely exists!

Proving the existence of periodic solutions

The Poincaré–Bendixson theorem provides a way of *proving* that a plane autonomous system has a periodic solution even when that solution cannot be found explicitly. If a phase path can be found that cannot escape from some bounded domain $\mathcal{D}$ of the phase plane, and if $\mathcal{D}$ contains no equilibrium points, then Poincaré–Bendixson implies that the phase path must either be a **closed loop** or tend to a **limit cycle**. In either case, the system must have a **periodic solution** lying in $\mathcal{D}$. The method is illustrated by the following examples.

Example 8.3 *Proving existence of a limit cycle*

Prove that the autonomous system of ODEs

$$\dot{x} = x - y - (x^2 + y^2)x,$$
$$\dot{y} = x + y - (x^2 + y^2)y,$$

has a limit cycle.

Solution

This system clearly has an equilibrium point at the origin $x = y = 0$, and a little algebra shows that there are no others. Although we have not proved this result, it is true that any periodic solution (simple closed loop) in the phase plane must have an equilibrium point lying *inside* it. In the present case, it follows that, if a periodic solution exists, then it *must* enclose the origin. This suggests taking the domain $\mathcal{D}$ to be the annular region between two circles centred on the origin.

It is convenient to express the system of equations in **polar coordinates** r, θ. The transformed equations are (see Problem 8.5)

$$\dot{r} = \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}, \qquad \dot{\theta} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2},$$

where $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$. In the present case, the polar equations take the simple form

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = 1.$$

* This actually describes a *stable* limit cycle, which is the only kind likely to be observed.

These equations can actually be solved explicitly, but, in order to illustrate the method, we will make no use of this fact. Let $\mathcal{D}$ be the annular domain $a < r < b$, where $0 < a < 1$ and $b > 1$. On the circle $r = b$, $\dot{r} = b(1 - b^2) < 0$. Thus a phase point that starts anywhere on the outer boundary $r = b$ *enters* the domain $\mathcal{D}$. Similarly, on the circle $r = a$, $\dot{r} = a(1 - a^2) > 0$ and so a phase point that starts anywhere on the inner boundary $r = a$ also *enters* the domain $\mathcal{D}$. It follows that *any phase path that starts in the annular domain $\mathcal{D}$ can never leave*. Since $\mathcal{D}$ is a *bounded* domain with *no equilibrium points* within it or on its boundaries, it follows from Poincaré–Bendixson that any such path must either be a simple closed loop or tend to a limit cycle. In either case, the system must have a **periodic solution** lying in the annulus $a < r < b$.

We can say more. Phase paths that begin on either *boundary* of $\mathcal{D}$ enter $\mathcal{D}$ and can never leave. These phase paths cannot close themselves (that would mean leaving $\mathcal{D}$) and so can only tend to a limit cycle. It follows that the system must have (at least one) **limit cycle** lying in the domain $\mathcal{D}$. [The explicit solution shows that the circle $r = 1$ is a limit cycle and that there are no other periodic solutions.] ■

Not all examples are as straightforward as the last one. Often, considerable ingenuity has to be used to find a suitable domain $\mathcal{D}$. In particular, the boundary of $\mathcal{D}$ cannot always be composed of circles. Most readers will find our second example rather difficult!

Example 8.4 *Rayleigh's equation has a limit cycle*

Show that **Rayleigh's equation**

$$\ddot{x} + \epsilon\dot{x}\left(\dot{x}^2 - 1\right) + x = 0,$$

has a limit cycle for any *positive* value of the parameter ϵ.

Solution

Rayleigh's equation arose in his theory of the bowing of a violin string. In the context of particle oscillations however, it corresponds to a simple harmonic oscillator with a strange damping term. When $|\dot{x}| > 1$, we have ordinary (positive) damping and the motion decays. However, when $|\dot{x}| < 1$, we have *negative damping* and the motion grows. The possibility arises then of a periodic motion which is positively damped on some parts of its cycle and negatively damped on others. Somewhat surprisingly, this actually exists.

Rayleigh's equation is equivalent to the autonomous system of ODEs

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -x - \epsilon v\left(v^2 - 1\right), \end{aligned} \tag{8.20}$$

for which the only equilibrium position is at $x = v = 0$. It follows that, if there is a periodic solution, then it must enclose the origin. At first, we proceed as in the first example. In polar form, the equations (8.20) become

$$\begin{aligned} \dot{r} &= \quad -\epsilon r \sin^2\theta\left(r^2\sin^2\theta - 1\right), \\ \dot{\theta} &= -1 - \epsilon\sin^2\theta\left(r^2\sin^2\theta - 1\right). \end{aligned} \tag{8.21}$$

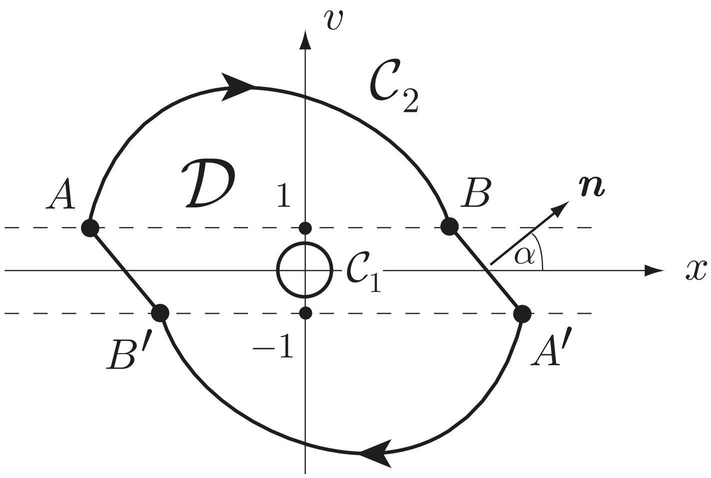

FIGURE 8.6 A suitable domain $\mathcal{D}$ to show that Rayleigh's equation has a limit cycle.

Let $r = c$ be a circle with centre at the origin and radius less than unity. Then $\dot{r} > 0$ everywhere on $r = c$ except at the two points $x = \pm c$, $v = 0$, where it is zero. Hence, except for these two points, we can deduce that a phase point that starts on the circle $r = c$ *enters* the domain $r > c$. Fortunately, these exceptional points can be disregarded. It does not matter if there are a *finite* number of points on $r = c$ where the phase paths go the 'wrong' way, since this provides only a *finite* number of escape routes! The circle $r = c$ thus provides a suitable *inner* boundary $\mathcal{C}_1$ of the domain $\mathcal{D}$.

Sadly, one cannot simply take a large circle to be the outer boundary of $\mathcal{D}$ since $\dot{r}$ has the wrong sign on those segments of the circle that lie in the strip $-1 < v < 1$. This allows any number of phase paths to escape and so invalidates our argument. However, this does not prevent us from choosing a boundary of a different shape. A suitable outer boundary for $\mathcal{D}$ is the contour $\mathcal{C}_2$ shown in Figure 8.6. This contour is made up from four segments. The first segment AB is part of an *actual phase path* of the system which starts at $A(-a, 1)$ and continues as far as $B(b, 1)$. The form of this phase path can be deduced from equations (8.21). When $v(= r\sin\theta) > 1$, $\dot{r} < 0$ and $\dot{\theta} < -1$, so that the phase point moves *clockwise* around the origin with r decreasing. In particular, B must be closer to the origin than A so that $b < a$, as shown. Similarly, the segment $A'B'$ is part of a second actual phase path that begins at $A'(a, -1)$. Because of the symmetry of the equations (8.20) under the transformation $x \to -x$, $v \to -v$, this segment is just the reflection of the segment AB in the origin; the point B' is therefore $(-b, -1)$. The contour is closed by inserting the straight line segments BA' and $B'A$.

We will now show that, when $\mathcal{C}_2$ is made sufficiently large, it is a suitable outer boundary for our domain $\mathcal{D}$. Consider first the segment AB. Since this *is* a phase path, no other phase path may cross it (in either direction); the same applies to the segment $A'B'$. Now consider the straight segment BA'. Because $a > b$, the outward unit normal $\boldsymbol{n}$ shown in Figure 8.6 makes a *positive* acute angle α with the axis Ox. Now the 'phase plane velocity' of a phase point is

$$\dot{x}\boldsymbol{i} + \dot{v}\boldsymbol{j} = v\boldsymbol{i} - \left(\epsilon v(v^2 - 1) + x\right)\boldsymbol{j}$$

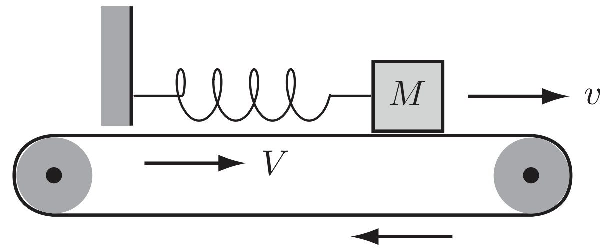

FIGURE 8.7 The body is supported by a rough moving belt and is attached to a fixed post by a light spring.

and the component of this 'velocity' in the $\boldsymbol{n}$-direction is therefore

$$\Big(v\boldsymbol{i} - \Big(\epsilon v(v^2-1) + x\Big)\,\boldsymbol{j}\Big) \cdot (\cos\alpha\,\boldsymbol{i} + \sin\alpha\,\boldsymbol{j})$$

$$\begin{aligned} &= v\cos\alpha - \sin\alpha\Big(\epsilon v(v^2-1) + x\Big) \\ &= -x\sin\alpha + v\Big(\cos\alpha + \epsilon\sin\alpha(1-v^2)\Big) \\ &< -b\sin\alpha + (1+\epsilon), \end{aligned}$$

for (x, v) on BA'. We wish to say that this expression is negative so that phase points that begin on BA' *enter* the domain $\mathcal{D}$. This is true if the contour $\mathcal{C}_2$ is made *large* enough. If we let a tend to infinity, then b also tends to infinity and α tends to $\pi/2$. It follows that, whatever the value of the parameter ϵ, we can make $b\sin\alpha > (1+\epsilon)$ by taking a large enough. A similar argument applies to the segment $B'A$. Thus the contour $\mathcal{C}_2$ is a suitable *outer* boundary for the domain $\mathcal{D}$. It follows that *any phase path that starts in the domain $\mathcal{D}$ enclosed by $\mathcal{C}_1$ and $\mathcal{C}_2$ can never leave*. Since $\mathcal{D}$ is a *bounded* domain with *no equilibrium points* within it or on its boundaries, it follows from Poincaré–Bendixson that any such path must either be a simple closed loop or tend to a limit cycle. In either case, Rayleigh's equation must have a **periodic solution** lying in $\mathcal{D}$.

We can say more. Phase paths that begin on either of the *straight* segments of the outer boundary $\mathcal{C}_2$ enter $\mathcal{D}$ and can never leave. These phase paths cannot close themselves (that would mean leaving $\mathcal{D}$) and so can only tend to a limit cycle. It follows that Rayleigh's equation must have (at least one) **limit cycle** lying in the domain $\mathcal{D}$. [There is in fact only one.] ■

A realistic mechanical system with a limit cycle

Finding *realistic* mechanical systems that exhibit limit cycles is not easy. Driven oscillations are eliminated by the requirement that the system be autonomous. Undamped oscillators have bounded *periodic* motions, and the introduction of damping causes the motions to die away to zero, not to a limit cycle. In order to keep the motion going, the system needs to be *negatively damped* for part of the time. This is an unphysical requirement, but it can be simulated in a physically realistic system as follows.

Consider the system shown in Figure 8.7. A block of mass M is supported by a rough horizontal belt and is attached to a fixed post by a light linear spring. The belt is made to move with constant speed V. Suppose that the motion takes place in a straight line and that $x(t)$ is the extension of the spring beyond its natural length at time t. Then the

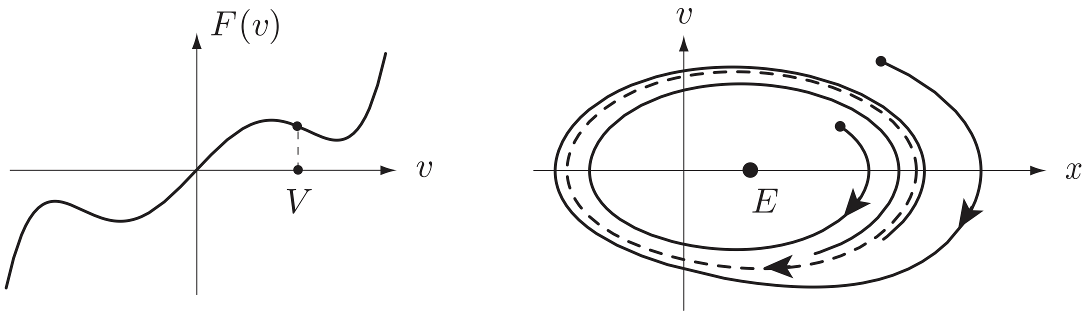

FIGURE 8.8 **Left**: The form of the frictional resistance function $G(v)$. **Right**: The limit cycle in the phase plane; E is the unstable equilibrium point.

equation of motion of the block is

$$M\frac{dv}{dt} = -M\Omega^2 x - F(v - V),$$

where $v = dx/dt$, $M\Omega^2$ is the spring constant, and $F(v)$ is the frictional force that the belt *would* exert on the block if the block had velocity v and the belt were *at rest*; in the actual situation, the argument v is replaced by the relative velocity $v - V$. The function $F(v)$ is supposed to have the form shown in Figure 8.8 (left). Although this choice is unusual ($F(v)$ is not an increasing function of v for all v), it is *not* unphysical!

Under the above conditions, the block has an equilibrium position at $x = F(V)/(M\Omega^2)$. The linearised equation for small motions near this equlibrium position is given by

$$M\frac{d^2x'}{dt^2} = -M\Omega^2 x' - F'(V)\frac{dx'}{dt},$$

where x' is the displacement of the block from the equilibrium position. If we select the belt velocity V so that $F'(V)$ is negative (as shown in Figure 8.8 (left)), then the *effective* damping is negative and small motions will grow. The equilibrium position is therefore *unstable*; oscillations of the block about the equlibrium position then do not die out, but instead tend to a **limit cycle**. This limit cycle is shown in Figure 8.8 (right). The formal proof that such a limit cycle exists is similar to that for Rayleigh's equation. Indeed, this system is essentially Rayleigh's model for the bowing of a violin string, where the belt is the bow, and the block is the string.

Chaotic motions

Another important conclusion from Poincaré–Bendixson is that *no bounded motion of a plane autonomous system can exhibit chaos*. The phase point cannot just wander about in a bounded region of the phase plane for ever. It must either close itself, terminate at an equilibrium point, or tend to a limit cycle and none of these motions is chaotic. In particular, no bounded motion of an undriven non-linear oscillator can be chaotic. As we will see in the next section however, the *driven* non-linear oscillator (a non-autonomous system) *can* exhibit bounded chaotic motions.

It should be remembered that Poincaré–Bendixson applies only to the bounded motion of *plane* autonomous systems. If the phase space has dimension three or more, then other motions, including chaos, are possible.

8.5 DRIVEN NON-LINEAR OSCILLATIONS

Suppose that we now introduce damping and a **harmonic driving force** into equation (8.2). This gives

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \Omega^2 x + \Lambda x^3 = F_0 \cos pt, \tag{8.22}$$

which is known as **Duffing's equation**.

The presence of the driving force $F_0 \cos pt$ makes this system *non-autonomous*. The behaviour of non-autonomous systems is considerably more complex than that of autonomous systems. Phase space is still a useful aid in *depicting* the motion of the system, but little can be said about the general behaviour of the phase paths. In particular, phase paths can cross each other any number of times, and Poincaré–Bendixson does not apply. Our treatment of driven non-linear oscillations is therefore restricted to perturbation theory.

In view of the large number of parameters, it is sensible to non-dimensionalise equation (8.22). The *dimensionless displacement* X is defined by $x = (F_0/p^2)X$ and the *dimensionless time* s by $s = pt$. The function $X(s)$ then satisfies the dimensionless equation

$$X'' + \left(\frac{k}{p}\right)X' + \left(\frac{\Omega}{p}\right)^2 X + \epsilon X^3 = \cos s, \tag{8.23}$$

where the dimensionless parameter ϵ is defined by

$$\epsilon = \frac{F_0^2 \Lambda}{\Omega^6}. \tag{8.24}$$

When $\epsilon = 0$, equation (8.23) reduces to the linear problem. This suggests that, when ϵ is small, we may be able to find approximate solutions by perturbation theory. The linear problem always has a periodic solution for X (the driven motion) that is harmonic with period 2π. Proving the existence of **periodic solutions** of Duffing's equation is an interesting and difficult problem. Here we address this problem for the case in which ϵ is small, a regular perturbation on the linear problem. To simplify the working we will suppose that damping is absent; the general features of the solution remain the same. The governing equation (8.23) then simplifies to

$$X'' + \left(\frac{\Omega}{p}\right)^2 X + \epsilon X^3 = \cos s. \tag{8.25}$$

Initial conditions do not come into this problem. We are simply seeking a family of solutions $X(s,\epsilon)$, parametrised by ϵ, that are (i) periodic, and (ii) reduce to the linear solution when $\epsilon = 0$. We need to consider first the **periodicity** of this family of solutions. In the non-linear problem, we have no right to suppose that the angular frequency of the driven motion is equal to that of the driving force, as it is in the linear problem; it could depend on ϵ. However, suppose that the driving force has minimum period τ_0 and that a family of solutions $X(s,\epsilon))$ of equation (8.25) exists with minimum period $\tau\ (= \tau(\epsilon))$. Then, since the derivatives and powers of X also have period τ, it follows that the left side of equation (8.25) must have period τ. The right side however has period τ_0 and this is known to be the minimum period. It follows that τ *must be an integer multiple of* τ_0; note that τ is not compelled to be *equal* to τ_0.* However, in the present case, the period $\tau(\epsilon)$ is supposed to be a *continuous* function of ϵ with $\tau = \tau_0$ when $\epsilon = 0$. It follows that the only possibility is that $\tau = \tau_0$ for all ϵ. Thus *the period of the driven motion is independent of* ϵ *and is equal to the period of the driving force*. This argument leaves open the possibility that other driven motions may exist that have periods that are integer multiples of τ_0. However, even if they exist, they cannot occur in our perturbation scheme.

We therefore expand $X(s,\epsilon)$ in the **perturbation series**

$$X(t,\epsilon) = X_0(t) + \epsilon X_1(t) + \epsilon^2 X_2(t) + \cdots, \tag{8.26}$$

and seek a solution of equation (8.25) that has period 2π. It follows that the expansion functions $X_0(s)$, $X_1(s)$, $X_2(s)$, ... must also have period 2π. If we now substitute this series into the equation (8.25) and equate coefficients of powers of ϵ, we obtain a succession of ODEs the first two of which are as follows:

From coefficients of ϵ^0:

$$X_0'' + \left(\frac{\Omega}{p}\right)^2 X_0 = \cos s. \tag{8.27}$$

From coefficients of ϵ^1:

$$X_1'' + \left(\frac{\Omega}{p}\right)^2 X_1 = -X_0^3. \tag{8.28}$$

For $p \neq \Omega$, the general solution of the zero order equation (8.27) is

$$X_0 = \left(\frac{p^2}{\Omega^2 - p^2}\right)\cos s + A\cos(\Omega s/p) + B\sin(\Omega s/p),$$

where A and B are arbitrary constants. Since X_0 is known to have period 2π, it follows that A and B must be zero unless Ω is an integer multiple of p; we will assume this is *not*

* The fact that τ is the minimum period of X does not *neccessarily* make it the minimum period of the left side of equation (8.25).

the case. Then the required solution of the **zero order** equation is

$$X_0 = \left(\frac{p^2}{\Omega^2 - p^2}\right)\cos s. \tag{8.29}$$

The **first order** equation (8.28) can now be written

$$\begin{aligned} X_1'' + \left(\frac{\Omega}{p}\right)^2 X_1 &= -\left(\frac{p^2}{\Omega^2 - p^2}\right)^3 \cos^3 s \\ &= -\left(\frac{p^6}{4(\Omega^2 - p^2)^3}\right)(3\cos s + \cos 3s), \end{aligned} \tag{8.30}$$

on using the trigonometric identity $\cos 3s = 4\cos^3 s - 3\cos s$. Since Ω/p is not an integer, the only solution of this equation that has period 2π is

$$X_1 = -\left(\frac{p^8}{4(\Omega^2 - p^2)^3}\right)\left(\frac{3\cos s}{\Omega^2 - p^2} + \frac{\cos 3s}{\Omega^2 - 9p^2}\right). \tag{8.31}$$

Results

When ϵ $(= F_0^3\Lambda/p^6)$ is small, the **driven response** of the Duffing equation (8.22) (with $k = 0$) is given by

$$x = \frac{F_0}{\Omega^2 - p^2}\left[\cos pt - \left(\frac{3p^6 \cos pt}{(\Omega^2 - p^2)^3} + \frac{p^6 \cos 3pt}{(\Omega^2 - p^2)^2(\Omega^2 - 9p^2)}\right)\epsilon + O\left(\epsilon^2\right)\right]. \tag{8.32}$$

This is the *approximate solution correct to the first order in the small parameter* ϵ. More terms can be obtained in a similar way but this is best done with computer assistance.

The most interesting feature of this formula is the behaviour of the first order correction term when Ω is close to $3p$, which suggests the existence of a *super-harmonic resonance* with frequency $3p$. Similar 'resonances' occur in the higher terms at the frequencies $5p, 7p, \ldots$, and are caused by the presence of the non-linear term Λx^3. It should not however be concluded that large amplitude responses occur at these frequencies.* The critical case in which $\Omega = 3p$ is solved in Problem 8.14 and reveals no infinities in the response.

Sub-harmonic responses and chaos

We have so far left open the interesting question of whether a driving force with minimum period τ can excite a **subharmonic response**, that is, a response whose minimum period is

* This is a subtle point. Like all power series, perturbation series have a certain 'radius of convergence'. When *all* the terms of the perturbation series are included, ϵ is resticted to some range of values $-\epsilon_0 < \epsilon < \epsilon_0$. What seems to happen when Ω approaches $3p$ is that ϵ_0 approaches zero so that the first order correction term never actually gets large.

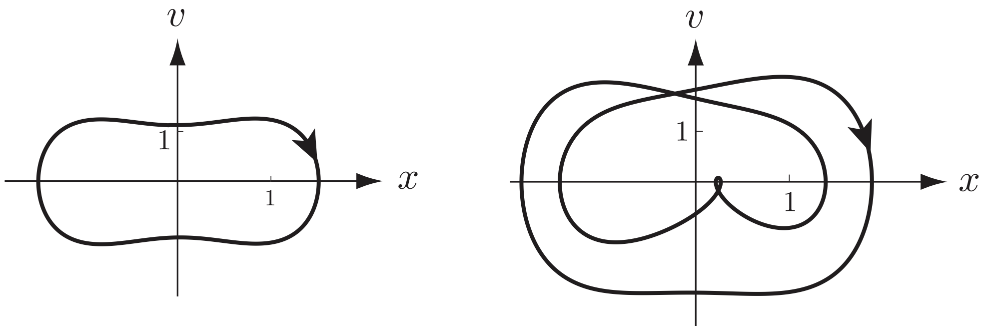

FIGURE 8.9 Two different periodic responses to the same driving force. **Left**: A response of period 2π, **Right**: A sub-harmonic response of period 4π.

an *integer multiple* of τ. This is certainly not possible in the linear case, where the driving force and the induced response always have the same period. One way of investigating this problem would be to expand the (unknown) response $x(t)$ as a Fourier series and to substitute this into the left side of Duffing's equation. One would then require all the odd numbered terms to magically cancel out leaving a function with period 2τ. Unlikely though this may seem, it can happen! *There are ranges of the parameters in Duffing's equation that permit a sub-harmonic response*. Indeed, it is possible for the *same* set of parameters to allow more than one periodic response. Figure 8.9 shows two different periodic responses of the equation $d^2x/dt^2 + kdx/dt + x^3 = A\cos t$, each corresponding to $k = 0.04$, $A = 0.9$. One response has period 2π while the other is a **subharmonic response** with period 4π. Which of these is the steady state response depends on the initial conditions. It is also possible for the motion to be **chaotic** with no steady state ever being reached, even though damping is present.

Problems on Chapter 8

Answers and comments are at the end of the book.

Harder problems carry a star ($*$).

Periodic oscillations: Lindstedt's method

8.1 A non-linear oscillator satisfies the equation

$$\left(1 + \epsilon x^2\right)\ddot{x} + x = 0,$$

where ϵ is a small parameter. Use Linstedt's method to obtain a two-term approximation to the oscillation frequency when the oscillation has unit amplitude. Find also the corresponding two-term approximation to $x(t)$. [You will need the identity $4\cos^3 s = 3\cos s + \cos 3s$.]

8.2 A non-linear oscillator satisfies the equation

$$\ddot{x} + x + \epsilon x^5 = 0,$$

where ϵ is a small parameter. Use Linstedt's method to obtain a two-term approximation to the oscillation frequency when the oscillation has unit amplitude. [You will need the identity $16\cos^5 s = 10\cos s + 5\cos 3s + \cos 5s$.]

8.3 ***Unsymmetrical oscillations*** A non-linear oscillator satisfies the equation

$$\ddot{x} + x + \epsilon x^2 = 0,$$

where ϵ is a small parameter. Explain why the oscillations are unsymmetrical about $x = 0$ in this problem.

Use Linstedt's method to obtain a two-term approximation to $x(t)$ for the oscillation in which the *maximum* value of x is unity. Deduce a two-term approximation to the *minimum* value achieved by $x(t)$ in this oscillation.

8.4* ***A limit cycle by perturbation theory*** Use perturbation theory to investigate the limit cycle of **Rayleigh's equation**, taken here in the form

$$\ddot{x} + \epsilon\left(\tfrac{1}{3}\dot{x}^2 - 1\right)\dot{x} + x = 0,$$

where ϵ is a small positive parameter. Show that the zero order approximation to the limit cycle is a circle and determine its centre and radius. Find the frequency of the limit cycle correct to order ϵ^2, and find the function $x(t)$ correct to order ϵ.

Phase paths

8.5 ***Phase paths in polar form*** Show that the system of equations

$$\dot{x}_1 = F_1(x_1, x_2, t), \qquad \dot{x}_2 = F_2(x_1, x_2, t)$$

can be written in polar coordinates in the form

$$\dot{r} = \frac{x_1 F_1 + x_2 F_2}{r}, \qquad \dot{\theta} = \frac{x_1 F_2 - x_2 F_1}{r^2},$$

where $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$.

A dynamical system satisfies the equations

$$\begin{aligned}\dot{x} &= -x + y,\\ \dot{y} &= -x - y.\end{aligned}$$

Convert this system into polar form and find the polar equations of the phase paths. Show that every phase path encircles the origin infinitely many times in the clockwise direction. Show further that every phase path terminates at the origin. Sketch the phase diagram.

8.6 A dynamical system satisfies the equations

$$\begin{aligned}\dot{x} &= x - y - (x^2 + y^2)x,\\ \dot{y} &= x + y - (x^2 + y^2)y.\end{aligned}$$

Convert this system into polar form and find the polar equations of the phase paths that begin in the domain $0 < r < 1$. Show that all these phase paths spiral anti-clockwise and tend to the limit cycle $r = 1$. Show also that the same is true for phase paths that begin in the domain $r > 1$. Sketch the phase diagram.

8.7 A damped linear oscillator satisfies the equation

$$\ddot{x} + \dot{x} + x = 0.$$

Show that the polar equations for the motion of the phase points are

$$\dot{r} = -r \sin^2 \theta, \qquad \dot{\theta} = -\left(1 + \tfrac{1}{2} \sin 2\theta\right).$$

Show that every phase path encircles the origin infinitely many times in the clockwise direction. Show further that these phase paths terminate at the origin.

8.8 A non-linear oscillator satisfies the equation

$$\ddot{x} + \dot{x}^3 + x = 0.$$

Find the polar equations for the motion of the phase points. Show that phase paths that begin within the circle $r < 1$ encircle the origin infinitely many times in the clockwise direction. Show further that these phase paths terminate at the origin.

8.9 A non-linear oscillator satisfies the equation

$$\ddot{x} + \left(x^2 + \dot{x}^2 - 1\right) \dot{x} + x = 0.$$

Find the polar equations for the motion of the phase points. Show that any phase path that starts in the domain $1 < r < \sqrt{3}$ spirals clockwise and tends to the limit cycle $r = 1$. [The same is true of phase paths that start in the domain $0 < r < 1$.] What is the period of the limit cycle?

8.10 ***Predator–prey*** Consider the symmetrical predator–prey equations

$$\dot{x} = x - xy, \qquad \dot{y} = xy - y,$$

where $x(t)$ and $y(t)$ are positive functions. Show that the phase paths satisfy the equation

$$\left(xe^{-x}\right)\left(ye^{-y}\right) = A,$$

where A is a constant whose value determines the particular phase path. By considering the shape of the surface

$$z = \left(xe^{-x}\right)\left(ye^{-y}\right),$$

deduce that each phase path is a simple closed curve that encircles the equilibrium point at $(1, 1)$. Hence *every solution* of the equations is periodic! [This prediction can be confirmed by solving the original equations numerically.]

Poincaré–Bendixson

8.11 Use Poincaré–Bendixson to show that the system

$$\dot{x} = x - y - (x^2 + 4y^2)x,$$
$$\dot{y} = x + y - (x^2 + 4y^2)y,$$

has a limit cycle lying in the annulus $\frac{1}{2} < r < 1$.

8.12 ***Van der Pol's equation*** Use Poincaré–Bendixson to show that Van der Pol's equation*

$$\ddot{x} + \epsilon\,\dot{x}\left(x^2 - 1\right) + x = 0,$$

has a limit cycle for any *positive* value of the constant ϵ. [The method is similar to that used for Rayleigh's equation in Example 8.4.]

Driven oscillations

8.13 A driven non-linear oscillator satisfies the equation

$$\ddot{x} + \epsilon\,\dot{x}^3 + x = \cos pt,$$

where ϵ, p are positive constants. Use perturbation theory to find a two-term approximation to the driven response when ϵ is small. Are there any restrictions on the value of p?

8.14 ***Super-harmonic resonance*** A driven non-linear oscillator satisfies the equation

$$\ddot{x} + 9x + \epsilon\,x^3 = \cos t,$$

where ϵ is a small parameter. Use perturbation theory to investigate the possible existence of a superharmonic resonance. Show that the zero order solution is

$$x_0 = \frac{1}{8}\left(\cos t + a_0 \cos 3t\right),$$

where the constant a_0 is a constant that is not known at the zero order stage.

By proceeding to the first order stage, show that a_0 is the unique real root of the cubic equation

$$3a_0^3 + 6a_0 + 1 = 0,$$

which is about -0.164. Thus, when driving the oscillator at this sub-harmonic frequency, the non-linear correction appears in the *zero order* solution. However, there are no infinities to be found in the perturbation scheme at this (or any other) stage.

Plot the graph of $x_0(t)$ and the path of the phase point $(x_0(t), x_0'(t))$.

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* After the extravagantly named Dutch physicist Balthasar Van der Pol (1889–1959). The equation arose in connection with the current in an electronic circuit. In 1927 Van der Pol observed what is now called *deterministic chaos*, but did not investigate it further.

Computer assisted problems

8.15 ***Linstedt's method*** Use computer assistance to implement Lindstedt's method for the equation

$$\ddot{x} + x + \epsilon x^3 = 0.$$

Obtain a three-term approximation to the oscillation frequency when the oscillation has unit amplitude. Find also the corresponding three-term approximation to $x(t)$.

8.16 ***Van der Pol's equation*** A classic non-linear oscillation equation that has a limit cycle is Van der Pol's equation

$$\ddot{x} + \epsilon \left(x^2 - 1\right) \dot{x} + x = 0,$$

where ϵ is a positive parameter. Solve the equation numerically with $\epsilon = 2$ (say) and plot the motion of a few of the phase points in the (x, v)-plane. All the phase paths tend to the limit cycle. One can see the same effect in a different way by plotting the solution function $x(t)$ against t.

8.17 ***Sub-harmonic and chaotic responses*** Investigate the *steady state* responses of the equation

$$\ddot{x} + k\dot{x} + x^3 = A \cos t$$

for various choices of the parameters k and A and various initial conditions. First obtain the responses shown in Figure 8.9 and then go on to try other choices of the parameters. Some very exotic results can be obtained! For various chaotic responses try $K = 0.1$ and $A = 7$.